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by

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SECOND-ORDER PERTURBATIONS DUE TO J_2, FOR A LOW-ECCENTRICITY EARTH-SATELLITE ORBIT

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SUMMARY

The perturbations of order J_2^2 in the orbital elements of an earth satellite are analysed by an elementary treatment that neglects terms of order $J_2^2 e$. The resulting expressions are combined into a triad of cylindrical polar coordinates, defined by a plane of fixed inclination and uniform rotation rate, since this leads to very simple formulae for perturbations in coordinates.

Mean orbital elements are required and are introduced in a general manner involving arbitrary constants. The normal choice of constants is such as to make both first-order and second-order perturbation formulae as compact as possible for the cylindrical coordinates, the second-order results being expressible as

$$\delta r = -\frac{1}{32} J_2^2 (R^2/a) f \{f \cos 4u + 2(26 - 31f) \cos 2u\} ,$$

$$\delta u' = -\frac{1}{32} J_2^2 (R/a)^2 f \{f \sin 4u - (19 - 20f) \sin 2u\}$$

and

$$\delta c = -\frac{3}{16} J_2^2 (R^2/a) f \sin 2i \sin 3u ,$$

where R is the earth's equatorial radius; a, i and u are the satellite's (mean) semi-major axis, inclination and argument of latitude, and f is $\sin^2 i$.

Other aspects of the J_2 'main problem' are considered.

Departmental Reference: Space 566

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1 INTRODUCTION

1.1 Previous papers on the subject

The description of the motion of an artificial satellite in the axi-symmetric field due to the lower-order zonal harmonics of the geopotential has been recognized as the 'main problem' in the theory of satellite orbits, and many solutions have been published. The first RAE paper[1] appeared in the same month as the first satellite and was succeeded by the theory which Merson[2] developed as a basis for the RAE's first proper orbit determination program[3]. Merson took this first theory as far as J_6 (results for an arbitrary J_ℓ were obtained by the present author[4]) and then[5] compared it with the theory of Kozai[6] (taken to J_4) that pioneered the orbit analysis of the Smithsonian Astrophysical Observatory. In a second theory[7], Merson developed the Kozai approach as the basis for a new orbit determination program, PROP, that[8,9] is still in use. The paper by Kozai may be regarded as one of the two classic American papers (its approach is summarized in a recent text-book[10]), the other being the paper of Brouwer[11] published at the same time. The Brouwer approach is followed in the book of Brouwer and Clemence[12], whilst one of the first text-books to give results (for J_2 only) was that of Sterne[13].

The early papers normally provided long-term solutions of the 'main problem', *ie* gave formulae for perturbations in mean orbital elements, to $O(J_2^2)$, where $J_\ell = O(J_2^2)$ if $\ell > 2$, but only took short-periodic perturbations to $O(J_2)$, neglecting them entirely for higher-order harmonics. It was found by Vinti[14-17] that, by use of spheroidal coordinates, a complete solution could be found for a field including J_2, but the higher-order even harmonics had to be powers of J_2 (*viz* $J_4 = -J_2^2$, $J_6 = J_2^3$, *etc*), so the general problem remained.

The first paper to take short-periodic perturbations to $O(J_2^2)$ seems to have been that of Petty and Breakwell[18]. Kozai, in a comprehensive development[19] to $O(J_2^2)$ for short-periodic perturbations and $O(J_2^3)$ for long-term perturbations, changed his approach to the method of von Zeipel that had been used by Brouwer. Aksnes[20] followed Hori[21] in basing his approach on the use of Lie series; by reference to a suitable intermediate orbit, he thereby obtained results that were equivalent to Kozai's but much more compact. Much more elementary (and therefore comprehensible) methods, just based on the planetary equations of Lagrange, were adopted by the French, the J_2^2 short-periodic perturbations being given by Bretagnon[22] and the J_2^3 long-term effects by Berger[23].

Solution of the J_2 problem to get a further order - short-periodic perturbations to $O(J_2^3)$ and long-term effects to $O(J_2^4)$ - was achieved by

Deprit and Rom[24] by the application of computer algebra to a method based on Lie series. Their results are only valid for moderately small values of the orbital eccentricity, e , since they are expressed as truncated power series in e . Techniques based on computer algebra are obviously very powerful, and two papers have recently been published that give main-problem solutions involving vast numbers of terms. Berger and Walch[25] cover harmonics as far as J_7 ; they only go to second order in short-periodic perturbations and (basically) third order in long-term effects, but this means the inclusion of secular terms in $J_5 J_7/J_2$, for example, that originate as fourth-order couplings between J_5 and J_7 , so that a large number of combinations are involved. Kinoshita[26] covers only the classical main-term harmonics, *ie* to J_4 only, but goes to third order for short-periodic perturbations and to fourth order for long-term effects; the perturbation method of Hori[21] is adopted and the resulting accuracy, for satellite motion in a low-eccentricity orbit, is better than 1 cm over a month.

Much of the literature cited is extremely difficult to follow, even for a reader with considerable experience of orbital analysis, mainly because of the extreme sophistication of the methods used, but probably also because of unfamiliarity with the notation. Without understanding the methods it is difficult to interpret the results - to pick out (say) a dominant set of terms for $O(J_2^2)$ perturbations for a near-circular orbit becomes virtually impossible. This is particularly true if expansions are given in terms of a quantity η , defined as $\sqrt{1 - e^2}$ (this quantity is denoted by q in the present paper).

1.2 The present paper

In many applications, involving satellites in orbits of low eccentricity, there are significant perturbations of order J_2^2 , including terms of short period, but terms in $J_2^2 e$ are entirely negligible. The significant terms may be obtained quite easily, by an elementary iteration on Lagrange's planetary equations, if a first-order solution to $O(J_2 e)$ is already available. Furthermore, when an appropriate set of non-singular elements is used, the results can be expressed very compactly. This is the approach of the present paper, the particular set of non-singular elements being (as defined in section 2.1) a, i, Ω, ξ, η and U .

The paper makes two contributions to the solution of the 'main problem', and discusses various aspects of the philosophy of perturbation theory. The first contribution concerns the derivation of formulae for perturbations in position, not merely orbital elements. If the satellite's position can be

expressed by three suitable coordinates, then the number of perturbation formulae required is immediately halved. (In certain applications, for example the processing of Doppler data, formulae for three velocity components will also be required, but these can be obtained from the position formulae, essentially by differentiation.) The reduction will be all the greater if the formulae for position are simpler than those for certain of the elements, as is clear from Kozai's original paper[6]. By introducing r (geocentric distance) and u (argument of latitude), Kozai was able to reduce long expressions for the (first-order) short-periodic perturbations in a, e, ω and M (defined here in section 2.1) into compact expressions for the perturbations in r and u; the expressions for the perturbations in i and Ω are much shorter and he left these alone. In developing the Kozai approach for use with PROP, Merson[7] presented six perturbation formulae that are extremely compact and totally free of singularity, and the present author modified[9] them slightly to reduce non-linear effects.

There is a different way to develop the Kozai approach, however, and in a recent study[27] of the perturbations of a Navstar satellite due to a general harmonic J_{lm} I presented results in what I called the 'RLC system' of mutually orthogonal directions, where R refers to r (as above), L to an along-track quantity ℓ that only exists at the differential level (and is such that $\delta\ell$ is the product of r with one of the six perturbations of Merson, *viz* $\delta u + \delta\Omega \cos i$), and C to a cross-track displacement (such that δc is the product of r with $\delta i \sin u - \delta\Omega \sin i \cos u$). Ref 27 does not cover J_2^2 effects, however, and the RLC system is not really suitable for effects beyond the first order in any J, because an integral ℓ is not defined; but it is a small step - taken here in section 2.5 - to a system that <u>is</u> suitable. The system amounts to a set of cylindrical polar coordinates defined in reference to a steadily rotating mean orbital plane, and the formulae for second-order short-periodic perturbations in these coordinates constitute the main results of this paper.

The paper's other contribution to the main problem relates to the definition of 'mean' orbital elements, and hence to the 'mean orbital plane' just referred to. The concept of a mean element underlies all approaches to the main problem, but there is an ambiguity in every such element in that the short-periodic perturbation (which, added to the mean element, gives the uniquely defined osculating element) is arbitrary by any quantity that is <u>free</u> of short-periodic perturbation. Thus if ζ is one of the six orbital elements and

$$\zeta_{osc} = \bar{\zeta} + J_2 \zeta_1 + J_2^2 \zeta_2 + J_2^3 \zeta_3 + \ldots .$$

(we later make such expansions in terms of a quantity $\bar{K}$, related to J_2 , rather than J_2 itself), then ζ_j is arbitrary to the extent of a quantity that could conveniently be denoted by $k_{j\zeta}$ and is itself a function of the six mean elements. Furthermore, if $j > 1$, each ζ_j is necessarily a function of the $k_{j\zeta}$ for smaller j .

Most papers on the main problem do not explain how their mean elements are defined, nor indeed admit that there is any problem. This makes it difficult to compare the results of different theories, and - worse - hard to interpret the published elements of actual satellites. The approach of the present paper is to face up to the problem by introducing the constant-like k-quantities explicitly into formulae, so that general results can be given without a pre-empted interpretation of particular mean elements. Individual first-order k_ζ (effectively the $k_{1\zeta}$ with appropriate 'scaling') are introduced in section 3 (subsequent to further discussion of mean elements in section 2.4), the most important being k_a, k_i and k_n .

There are three possible criteria which can underlie the choice of the k-quantities and which (tacitly at least) are applied in the literature. First, the k's may be chosen so as to make the expressions for positional perturbations as simple as possible; this is the philosophy adopted here. Secondly, they may be chosen such that time averages of the short-periodic perturbations are strictly zero; this amounts to making the k's zero if perturbations are expressed in terms of mean anomaly (M) rather than true anomaly (v). Thirdly, the k's may be set to zero when the perturbations are expressed in terms of v ; higher-order formulae will then be simpler than with the second choice.

This brings us to another important point. Early formulations of the short-periodic perturbations (*eg* Refs 6 and 13) were in terms of v (or u , where $u - v = \omega$, the argument of perigee) and hence expressible in closed form. Later papers, and in particular those based on computer algebra, switched to M , for reasons that are not entirely clear, the result being expressions that are necessarily truncations of power series in e . The present paper, even though it truncates before the $O(J_2^2\,e)$ terms, uses v (or rather u); there is no obvious reason why, in terms of v , the main problem should not be solved to further order without a requirement for infinite series in e (but see also the remarks of Refs 24 and 28; the difficulty lies in evaluating $\int v\,dM$).

Section 9 of this paper is devoted to complete expressions for the second-order long-term variation of the orbital elements, divided into secular and long-periodic perturbations in the usual way. When $O(J_2^2\,e)$ effects can be neglected,

no long-periodic perturbations arise (the meaning of 'second-order' in connection with these perturbations is discussed), but (as already indicated) it makes sense to take long-term perturbations to a higher order than short-periodic perturbations, unless the satellite's motion is only to be represented for a matter of, say, at most a revolution or two. For completeness, also, the untruncated expressions for the first-order perturbations in positional coordinates are given in section 8, since section 3 takes them only to $O(J_2 e)$.

The results obtained for perturbations in orbital elements are compared with those given by Bretagnon[22], Berger and Walch[25], and Kinoshita[26], all discrepancies being resolved. The formulae for second-order positional perturbations have been checked by comparing computed results with those given by numerical integration.

2 BACKGROUND

2.1 Osculating elements

The standard osculating elements of an elliptic orbit are a (semi-major axis), e (eccentricity) , i (inclination) , Ω (right ascension of the node) , ω (argument of perigee) and M (mean anomaly) . Since e is here assumed to be small, we shall also require the non-singular elements ξ , η and U , defined by

$$\xi = e \cos \omega \ , \tag{1}$$

$$\eta = e \sin \omega \tag{2}$$

and

$$U = M + \omega \ . \tag{3}$$

The mean motion, n , though defined by

$$n^2 a^3 = \mu \ , \tag{4}$$

where μ is the earth's gravitational constant, will often be treated as an element in its own right. Finally, it is convenient to introduce f, h, p and q , where

$$f = \sin^2 i \ , \tag{5}$$

$$h = 1 - 1\tfrac{1}{2}f \ , \tag{6}$$

$$p = a(1 - e^2) \tag{7}$$

and

$$q = (1 - e^2)^{\frac{1}{2}} \ . \tag{8}$$

2.2 Assumed potential

As we are only concerned with the J_2 term of the earth's disturbing potential, U, the potential is taken as $\mu/r + U$, where

$$U = -\frac{\mu}{r} J_2 \left(\frac{R}{r}\right)^2 P_2(\sin\beta) \quad ; \tag{9}$$

here r is the radial distance of the satellite from the centre of the earth, β is its geocentric latitude, R is the earth's equatorial radius, and P_2 is the usual Legendre polynomial. Thus we are neglecting drag, lunisolar attraction, *etc*, as well as the other harmonics of the real geopotential.

To eliminate β, we introduce the argument of latitude, u, of the satellite, given by

$$u = v + \omega \quad , \tag{10}$$

where v is the true anomaly (see section 2.5); thus,

$$\sin\beta = \sin i \sin u$$

and hence

$$U = \tfrac{1}{4} \mu J_2 \frac{R^2}{r^3} \{2 - 3f\ (1 - \cos 2u)\} \quad . \tag{11}$$

It is easy to eliminate r, since

$$\frac{p}{r} = 1 + e\cos v \quad . \tag{12}$$

2.3 Lagrange's planetary equations

Rates of change of osculating elements may be expressed exactly by Lagrange's planetary equations[10,12,13], *viz*:

$$\dot{a} = \frac{2}{na}\frac{\partial U}{\partial M} , \tag{13}$$

$$\dot{e} = \frac{1}{na^2 e}\left\{q^2 \frac{\partial U}{\partial M} - q\frac{\partial U}{\partial \omega}\right\} , \tag{14}$$

$$\dot{i} = \frac{\operatorname{cosec} i}{na^2 q}\left\{\cos i \frac{\partial U}{\partial \omega} - \frac{\partial U}{\partial \Omega}\right\} , \tag{15}$$

$$\dot{\Omega} = \frac{\text{cosec } i}{na^2 q} \frac{\partial U}{\partial i} , \tag{16}$$

$$\dot{\omega} = \frac{1}{na^2} \left\{ \frac{q}{e} \frac{\partial U}{\partial e} - \frac{\cot i}{q} \frac{\partial U}{\partial i} \right\} \tag{17}$$

and

$$\dot{M} = n + \dot{\sigma} , \tag{18}$$

where

$$\dot{\sigma} = - \frac{1}{na^2} \left\{ \frac{q^2}{e} \frac{\partial U}{\partial e} + 2a \frac{\partial U}{\partial a} \right\} . \tag{19}$$

Equation (19) effectively defines σ as the 'modified mean anomaly at epoch' $(t = 0)$, on the basis that

$$M = \sigma + \int_0^t n \, dt . \tag{20}$$

It will be observed that $\dot{U}$, where U is given by (3), is not quite free of $\partial U/\partial e$, since $\partial U/\partial e$ is multiplied by different powers of q in (17) and (19). Thus though U is a better element than M for use when e is small, it can complicate matters, rather than simplify them, when a full analysis is required.

The partial derivatives required in Lagrange's equations can be obtained from (11), bearing in mind that

$$\frac{\partial v}{\partial e} = \frac{\sin v \, (2 + e \cos v)}{q^2}$$

and

$$\frac{\partial v}{\partial M} = \frac{a^2 q}{r^2} = \frac{(1 + e \cos v)^2}{q^3} .$$

It is also convenient to introduce, for general use throughout the paper, the quantity K , where

$$K = 1\tfrac{1}{2} \, J_2 \left(\frac{R}{p}\right)^2 , \tag{21}$$

so that we may re-express U as

$$U = \frac{1}{6} \frac{\mu K}{p} (2h + 3f \cos 2u)(1 + e \cos v)^3 . \tag{22}$$

(It has to be remembered, from time to time, that K is not constant; thus
100 $\partial K/\partial a$ - and less importantly $\partial K/\partial e$ - is not zero.)

Then exact expressions for the rates of change of the elements are:

$$\dot{a} = -\frac{Kna}{2q^5}\left(\frac{p}{r}\right)^4 \left\{4f \sin 2u + e\left[4h \sin v - f \sin (u + \omega) + 5f \sin (2u + v)\right]\right\} , \qquad \ldots\ldots (23)$$

$$\dot{e} = -\frac{Kn}{4q^3}\left(\frac{p}{r}\right)^3 \left\{4h \sin v + f \sin (u + \omega) + 7f \sin (2u + v) + \tfrac{1}{2}e\left[4h \sin 2v + 12f \sin 2u + 5f \sin 2(u + v) - f \sin 2\omega\right]\right\} , \qquad \ldots\ldots (24)$$

$$\dot{i} = -\frac{Kn \sin 2i}{2q^3}\left(\frac{p}{r}\right)^3 \sin 2u \quad , \qquad (25)$$

$$\dot{\Omega} = -\frac{Kn \cos i}{q^3}\left(\frac{p}{r}\right)^3 (1 - \cos 2u) \quad , \qquad (26)$$

$$e\,\dot{\omega} = \frac{Kn}{4q^3}\left(\frac{p}{r}\right)^3 \left\{4h \cos v - f \cos (u + \omega) + 7f \cos (2u + v) + \tfrac{1}{2}e\left[4h \cos 2v - 2\,(4 - 7f) \cos 2u + 5f \cos 2(u + v) + f \cos 2\omega + 2(6 - 7f)\right]\right\} \qquad \ldots\ldots (27)$$

and

$$e\,\dot{\sigma} = -\frac{Kn}{4q^2}\left(\frac{p}{r}\right)^3 \left\{4h \cos v - f \cos (u + \omega) + 7f \cos (2u + v) + \tfrac{1}{2}e\left[4h \cos 2v - 18f \cos 2u + 5f \cos 2(u + v) + f \cos 2\omega - 12h\right]\right\} . \qquad (28)$$

The last two equations are written with e-factors on the left-hand side to emphasise their singularity. The sum, $\dot{\sigma} + \dot{\omega}$, is free of singularity, and will be used in section 4.7, but as an exact combination it does not simplify, because of the difference in powers of q already remarked.

Exact expressions for $\dot{\xi}$ and $\dot{\eta}$ are easy to obtain, since

$$\dot{\xi} = \dot{e} \cos \omega - e\,\dot{\omega} \sin \omega \qquad (29)$$

and

$$\dot{\eta} = \dot{e} \sin \omega + e\,\dot{\omega} \cos \omega \quad . \qquad (30)$$

Thus

$$\dot{\xi} = -\frac{Kn}{4q^3}\left(\frac{p}{r}\right)^3 \Big\{(4h + f)\sin u + 7f \sin 3u$$
$$+ \tfrac{1}{2}e\big[3(4 - 5f)\sin\omega + (8 - 7f)\sin(u + v)$$
$$- (4 - 13f)\sin(2u + \omega) + 5f\sin(3u + v)\big]\Big\} \qquad (31)$$

and

$$\dot{\eta} = \frac{Kn}{4q^3}\left(\frac{p}{r}\right)^3 \Big\{(4h - f)\cos u + 7f\cos 3u$$
$$+ \tfrac{1}{2}e\big[(12 - 13f)\cos\omega - 5f\cos(u + v)$$
$$- (4 - 12f)\cos(2u + \omega) + 5f\cos(3u + v)\big]\Big\} \quad . \qquad (32)$$

2.4 Mean elements

Let ζ be an osculating element, *ie* it stands for any one of $a, e, i, \Omega,$ $\omega, \sigma, M, \xi, \eta$ and U. Its variation is made up of a long-term effect, not directly related to the orbital period (*ie* it is independent of u and v), and of a series of short-periodic terms (period a submultiple of the orbital period). The combination of the short-periodi: terms is the short-periodic perturbation, $\delta\zeta$, removal of which from ζ gives a mean element, $\bar{\zeta}$, which for most purposes (because its variation can be plotted easily and accurately over long periods of time) is more useful than the osculating e .t; thus

$$\zeta = \bar{\zeta} + \delta\zeta \quad . \qquad (33)$$

The variation of $\bar{\zeta}$ in general has two components, a purely secular effect, which is zero for several of the elements, and a long-periodic effect related to the secular variation of ω. Now, to the first order in J_2 there is no long-periodic effect (the meaning of 'order' in relation to long-periodic perturbations is discussed in section 9), since long-periodic effects are induced by the first-order variation of ω and hence only appear at higher-than-first order; at second order, though there is an effect, it is $O(J_2^2 e)$, which (until section 9) we are neglecting. Hence we may (normally) write

$$\bar{\zeta} = \bar{\zeta}_0 + \dot{\bar{\zeta}}\, t \quad , \qquad (34)$$

where $\dot{\bar{\zeta}}$ vanishes when ζ is a, e or i. It will sometimes be useful to write

$$\zeta = \bar{\zeta}_0 + \Delta\zeta \quad , \tag{35}$$

where, from (33) and (34)

$$\Delta\zeta = \dot{\bar{\zeta}}\, t + \delta\zeta \quad . \tag{36}$$

We shall be expressing $\delta\zeta$ as a polynomial (truncated to a quadratic) in $\bar{K}$, where $\bar{K}$ is K, as given by (21), with p replaced by $\bar{p}$. It is evident that the 'coefficient' of each 'term' of the polynomial is itself a function of the (mean) orbital elements, possibly involving many terms, and that such a function is arbitrary to the extent of a u-independent term, *ie* that mean elements are only defined to within selected 'constants'. Choice of a set of constants is a matter of convenience, and published papers do not always make clear what particular constants have been chosen, or the motivation for the authors' choice.

As indicated in section 1, the present paper generalizes the approach to mean elements by explicitly introducing arbitrary 'constants' (k_a, k_e *etc*) into the expressions, ζ_1, that occur as coefficients of $\bar{K}$ in the various $\delta\zeta$. At the second-order level, similarly, we meet k_{2a} *etc*.

2.5 Satellite position

The significance of a set of mean elements, $\bar{\zeta}$, is perhaps best understood by seeing how they can be used directly (in conjunction with perturbation formulae) to generate a satellite's position coordinates, *viz* x, y and z.

The standard algorithm for x, y and z, given the *osculating* elements a, e, i, Ω, ω and M, is as follows:

(i) the eccentric anomaly, E, is found by solving Kepler's equation

$$E - e \sin E = M \quad ; \tag{37}$$

(ii) the true anomaly, v, is found from one of the two equivalent formulae (apart from an ambiguity of quadrant in the first formula)

$$\tan v = \frac{q \sin E}{\cos E - e} \tag{38}$$

and

$$\tan \tfrac{1}{2}v = \left(\frac{1 + e}{1 - e}\right)^{\frac{1}{2}} \tan \tfrac{1}{2}E \quad ; \tag{39}$$

(iii) u is obtained from (10), and r from either (12) or the equivalent formula

$$r = a(1 - e\cos E) \quad ; \tag{40}$$

(iv) x, y and z are obtained from the double coordinate transformation expressed by the matrix formula

$$(x \quad y \quad z)^T = R_3(-\Omega)\, R_1(-i)\, (r\cos u \quad r\sin u \quad 0)^T \ , \tag{41}$$

where $R_j(\theta)$ describes rotation about the j*th* axis, so that

$$R_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \textit{etc} \quad ;$$

also T, in (41), denotes transposition.

There is, of course, a preliminary step given by

$$M = M_0 + nt \ , \tag{42}$$

if the mean anomaly is only available at epoch, though appropriate for use at time t, and it may be necessary to recover e, ω and M from non-singular elements ξ, η and U defined by equations (1) to (3).

Now suppose the starting point is a set of *mean* elements at epoch, with secular rates and short-periodic perturbations known. Then each ζ (osculating) can be obtained from (35) before operating the algorithm, but there is an alternative procedure which is more satisfactory since it requires only four δ-expressions instead of six. In this procedure, which was given by Kozai[6], quantities δr and δu, representing short-periodic perturbations in r and u, appear instead of δa, δe, $\delta\omega$ and δM. Steps (i) to (iii) of the standard algorithm now operate on the *mean* elements, leading in turn to $\bar{E}$, $\bar{v}$, $\bar{u}$ and $\bar{r}$, after which, as an additional part of step (iii), r and u are derived as $\bar{r} + \delta r$ and $\bar{u} + \delta u$ respectively. The operation of step (iv) is exactly as before.

But Kozai's modification of the standard algorithm for x, y and z can be extended, to its logical conclusion, so that only *three* δ-expressions are required. Two of the perturbed quantities are slightly modified forms of r and u, and
100 may thus be conveniently written r' and u', whilst the third represents the

displacement of the satellite from its 'mean orbital plane', *ie* from a plane of fixed inclination, $\bar{i}$, that rotates at a fixed rate, $\dot{\bar{\Omega}}$. This displacement is in the 'cross-track direction' and may be conveniently denoted by c , though we cannot distinguish the perturbation δc from c itself, since the unperturbed value is zero. The positive direction for c is such that the (positive) direction of orbital motion is given by the right-hand screw rule, the first-order formula for δc being in consequence

$$\delta c = r\,(\delta i \sin \bar{u} - \delta\Omega \sin \bar{i} \cos \bar{u}) \quad , \tag{43}$$

as given in Ref 27.

This approach gives a complete first-order representation of δi via δc , but only represents the $\delta\Omega \sin \bar{i}$ component of $\delta\Omega$. The other component has to be incorporated with δu , the combined effect being $\delta u'$ and given to first order by

$$\delta u' = \delta u + \delta\Omega \cos \bar{i} \quad . \tag{44}$$

The difference between r' and r is zero to first order.

The operation of step (iv) is different with this approach, since $\bar{i}$ and $\bar{\Omega}$, not i and Ω , must be used in the coordinate transformation. Thus (41) is replaced by the formula

$$(x \quad y \quad z)^T = R_3(-\bar{\Omega})\, R_1(-\bar{i})\, (r' \cos u' \quad r' \sin u' \quad c)^T \quad . \tag{45}$$

It is clear that

$$r'^2 + c^2 = r^2 \quad , \tag{46}$$

since they are equal to the same thing, *viz* $x^2 + y^2 + z^2$. It is from (46) that we see the first-order identity between r' and r ; if δc can be kept to $O(Ke, K^2)$, indeed, then $r' - r$ will be $O(K^2e^2, K^3e, K^4)$ and hence (for our purpose) entirely negligible. In section 3 we shall see that δc can be kept to $O(Ke, K^2)$ by appropriate choice of k_i and k_Ω .

The quantities r', u' and c form a set of cylindrical polar coordinates relative to the steadily rotating mean orbital plane. Thus the directions in which r' increases, u' increases and c increases are orthogonal, corresponding to the axis directions for the RLC system of coordinates previously used by the author[27]. The trouble with that system is that the 'L' coordinate is only defined differentially (as remarked in section 1) on the basis that

$$\delta\ell = r\,\delta u' \quad . \tag{47}$$

2.6 Formulae connecting mean and true anomaly

Formulae connecting M and v will be required throughout the paper. Suitable formulae are collected here for reference. They are valid for both osculating elements and mean elements.

The expression for the difference between M and v is known as 'the equation of the centre'. The following is an exact form of the equation, expressed as a double series of Bessel functions, as given by page 77 of Ref 12:

$$v = M + 2\sum_{j=1}^{\infty} \frac{\sin jM}{j}\left\{J_j(je) + \sum_{k=1}^{\infty}\left(\frac{e}{1+q}\right)^k \left[J_{j-k}(je) + J_{j+k}(je)\right]\right\} \quad . \tag{48}$$

This leads to

$$v = M + 2e \sin M + 1\tfrac{1}{4}e^2 \sin 2M - \tfrac{1}{12}e^3 (3 \sin M - 13 \sin 3M) + O(e^4) \quad , \tag{49}$$

of which the equivalent v-series result is

$$M = v - 2e \sin v + \tfrac{3}{4}e^2 \sin 2v - \tfrac{1}{3}e^3 \sin 3v + O(e^4) \quad . \tag{50}$$

From (49) we can derive formulae for sin v and cos v, *viz*

$$\sin v = \sin M + e \sin 2M - \tfrac{1}{8}e^2 (7 \sin M - 9 \sin 3M) - \tfrac{1}{6}e^3 (7 \sin 2M - 8 \sin 4M) + O(e^4) \tag{51}$$

and

$$\cos v = \cos M - e (1 - \cos 2M) - 1\tfrac{1}{8}e^2 (\cos M - \cos 3M) - 1\tfrac{1}{3}e^3 (\cos 2M - \cos 4M) + O(e^4) \quad . \tag{52}$$

Finally, we generalize (51) and (52) to O(e) expressions for the sine and cosine of $(jv + k\omega)$. Allowing T to denote either one of these functions, we have

$$T(jv + k\omega) = T(jM + k\omega) + je\, T\{(j+1)M + k\omega\} - je\, T\{(j-1)M + k\omega\} + O(e^2) \quad . \qquad \ldots\ldots \tag{53}$$

We can, of course, replace M by v , if desired, in the two je-terms. Also, by taking $j = k$, we can derive the relations between the sines and cosines of ku and kU .

3 FIRST-ORDER RESULTS

Complete first-order perturbation formulae may be expressed in closed form, since powers of e beyond the third do not arise. To integrate equations (23) to (28), (31) and (32) with respect to t, we transform to v as independent variable by use of the relation (for an unperturbed orbit)

$$\frac{dv}{dt} = \frac{n}{q^3}\left(\frac{p}{r}\right)^2 . \tag{54}$$

This leads to an expression for $d\zeta/dv$, for each element ζ, in terms of a power of (p/r) and trigonometric terms in v and ω. The power of (p/r) can be expanded by use of (12), after which the expression for $d\zeta/dv$ can be integrated with respect to v, all elements being held constant.

The results have been given, in various equivalent forms, by a number of authors, *eg* in Refs 10, 12 and 13. They are given again, in a compact form, here for completeness. Since the secular terms in Ω, ω and σ have argument v rather than t, they are not immediately separable from the short-periodic terms, whereas for M the separation of ΔM into $\dot{\bar{M}}t$ and δM is immediate.

The formulae, in which arbitrary 'constants' have been omitted (see section 2.4) and in which compactness is achieved largely by writing C_j for $\cos(jv + 2\omega)$ and S_j for $\sin(jv + 2\omega)$, are:-

$$\begin{aligned}\delta a = \tfrac{1}{24}Kaq^{-2}\Big\{24f\,C_2 &+ e\left[48h\cos v + 36f\,(C_1 + C_3)\right] \\ &+ e^2\left[24h\cos 2v + 18f\,(2C_2 + C_4)\right] \\ &+ e^3\left[4h\,(3\cos v + \cos 3v) + 3f\,(C_{-1} + 3C_1 + 3C_3 + C_5)\right]\Big\} ,\end{aligned} \tag{55}$$

$$\begin{aligned}\delta e = \tfrac{1}{48}K\Big\{48h\cos v + 4f\,(3C_1 + 7C_3) &+ 6e\left[4h\cos 2v + 10f\,C_2 + 3f\,C_4\right] \\ &+ e^2\left[4h\,(3\cos v + \cos 3v) + f\,(3C_{-1} + 33C_1 + 17\,C_3 + 3C_5)\right]\Big\} ,\end{aligned} \tag{56}$$

$$\delta i = \tfrac{1}{12}K\sin 2i\left\{3C_2 + e\left[3C_1 + C_3\right]\right\} , \tag{57}$$

$$\Delta\Omega = \tfrac{1}{6}K\cos i\left\{-6v + 3S_2 - e\left[6\sin v - 3S_1 - S_3\right]\right\} , \tag{58}$$

$$\Delta\omega = \tfrac{1}{48} K\, e^{-1} \Big\{ 48h \sin v - 4f\,(3S_1 - 7S_3) + 6e\,\big[4\,(4 - 5f)\,v + 4h \sin 2v - 2\,(2 - 5f)\,S_2 + 3f\,S_4\big] + e^2\,\big[6\,(14 - 17f) \sin v + 4h \sin 3v - 3f\,S_{-1} - 3\,(8 - 15f)\,S_1 - (8 - 19f)\,S_3 + 3f\,S_5\big]\Big\} \,, \tag{59}$$

$$\Delta\sigma = -\tfrac{1}{48} K\, e^{-1}\, q\, \Big\{ 48h \sin v - 4f\,(3S_1 - 7S_3) + 6e\,\big[-8h\,v + 4h \sin 2v - 6f\,S_2 + 3f\,S_4\big] + e^2\,\big[-60h \sin v + 4h \sin 3v - f\,(3S_{-1} + 51S_1 + 13S_3 - 3S_5)\big]\Big\} \tag{60}$$

and

$$\Delta M = n'\,t - \tfrac{1}{48} K\, e^{-1}\, q\, \Big\{ 48h \sin v - 4f\,(3S_1 - 7S_3) + 6e\,\big[4h \sin 2v + 3f\,S_4\big] - e^2\,\big[4h\,(3 \sin v - \sin 3v) + f\,(3S_{-1} + 15S_1 + S_3 - 3S_5)\big]\Big\} \,. \tag{61}$$

The 'mean mean motion', n', that appears in (61) is an absolute constant of the motion that is related to the energy integral as explained in Refs 2 and 4 (see also section 4.1). It is given by

$$n'^2\, a'^3 = \mu \,, \tag{62}$$

where the relation between a' and (osculating) a is

$$\frac{1}{a'} = \frac{1}{a}\left(1 + \frac{2aU}{\mu}\right) . \tag{63}$$

This relation is exact, not merely first-order in J_2. To first order - *cf* equation (114) to be given in section 4.1 - we have

$$a = a' + \delta a + \tfrac{1}{12} K\,(a^2/p)\,\{8h + 3e^2\,(4h + 3f \cos 2\omega)\} \,, \tag{64}$$

with δa given by (55), so that a' corresponds to a particular choice of mean element $\bar{a}$.

Two other comments are worth making about equations (55) to (61). First, the quantities on the right-hand sides should all be regarded as *mean*, so that strictly $\bar{K}$, $\bar{e}$, $\bar{v}$ should be written throughout; being only first-order equations, however, they are actually correct as they stand. Secondly, the presence of the q-factor in (60) and (61) makes a simple complete expression for δU impossible

(*cf* the corresponding remark in section 2.3); complete expressions for $\delta\xi$ and $\delta\eta$ are simple enough, however, but will not be given in this paper.

We now write down the perturbations, to $O(e)$ only, that we shall require in the sequel. For each element ζ we express $\bar{\dot{\zeta}}$ and $\delta\zeta$ in the form $\bar{K}\bar{\dot{\zeta}}_1$ and $\bar{K}\zeta_1$ respectively, except that we cannot define $\bar{\dot{M}}_1$ or $\bar{\dot{U}}_1$ in this way, as M and U are not constant for an unperturbed orbit. It is convenient to drop e, ω, σ and M in favour of ξ, η, U and n, and to introduce k-constants at appropriate points. Then the formulae required are:-

$$a_1 = \tfrac{1}{2}\bar{a}\left\{2(\bar{f}\cos 2\bar{u} + k_a) + \bar{e}\left[4\bar{h}\cos\bar{v} + 3\bar{f}\cos(\bar{u}+\bar{\omega}) + 3\bar{f}\cos(2\bar{u}+\bar{v})\right]\right\}, \quad \ldots\ldots (65)$$

$$i_1 = \tfrac{1}{12}\sin 2\bar{i}\left\{3(\cos 2\bar{u} + k_i) + \bar{e}\left[3\cos(\bar{u}+\bar{\omega}) + \cos(2\bar{u}+\bar{v})\right]\right\}, \quad (66)$$

$$\bar{\dot{\Omega}}_1 = -\bar{n}\cos\bar{i}, \quad (67)$$

$$\Omega_1 = \tfrac{1}{6}\cos\bar{i}\left\{3(\sin 2\bar{u} + k_\Omega) - \bar{e}\left[18\sin\bar{v} - 3\sin(\bar{u}+\bar{\omega}) - \sin(2\bar{u}+\bar{v})\right]\right\}, \quad \ldots\ldots (68)$$

$$\bar{\dot{\xi}}_1 = -\tfrac{1}{2}\bar{n}\bar{e}(4-5\bar{f})\sin\bar{\omega}, \quad (69)$$

$$\xi_1 = \tfrac{1}{24}\Big\{2\left[3(4\bar{h}+\bar{f})\cos\bar{u} + 7\bar{f}\cos 3\bar{u}\right] + 3\bar{e}\big[6(1-\bar{f})\cos(\bar{u}+\bar{v}) - 2(1-5\bar{f})\cos(2\bar{u}+\bar{\omega}) + 3\bar{f}\cos(3\bar{u}+\bar{v}) + k_\xi\big]\Big\}, \quad (70)$$

$$\bar{\dot{\eta}}_1 = \tfrac{1}{2}\bar{n}\bar{e}(4-5\bar{f})\cos\bar{\omega}, \quad (71)$$

$$\eta_1 = \tfrac{1}{24}\Big\{2\left[3(4\bar{h}-\bar{f})\sin\bar{u} + 7\bar{f}\sin 3\bar{u}\right] + 3\bar{e}\big[2(1-3\bar{f})\sin(\bar{u}+\bar{v}) - 2(1-5\bar{f})\sin(2\bar{u}+\bar{\omega}) + 3\bar{f}\sin(3\bar{u}+\bar{v}) + k_\eta\big]\Big\}, \quad \ldots\ldots (72)$$

$$\bar{\dot{U}} = n' + \bar{\dot{\omega}}, \quad \textit{ie}\quad n' + \tfrac{1}{2}\bar{K}\bar{n}(4-5\bar{f}), \quad (73)$$

$$U_1 = -\tfrac{1}{24}\Big\{6(2-5\bar{f})\sin 2\bar{u} - \bar{e}\big[6(26-33\bar{f})\sin\bar{v} - 3(4-9\bar{f})\sin(\bar{u}+\bar{\omega}) - (4-17\bar{f})\sin(2\bar{u}+\bar{v})\big]\Big\} \quad (74)$$

and

$$n_1 = -\tfrac{3}{4}\bar{n}\left\{2(\bar{f}\cos 2\bar{u} + k_n) + \bar{e}\left[4\bar{h}\cos\bar{v} + 3\bar{f}\cos(\bar{u}+\bar{\omega}) + 3\bar{f}\cos(2\bar{u}+\bar{v})\right]\right\}. \quad \ldots\ldots (75)$$

NB. These formulae are only valid to $O(e)$; also, k_a is redefined in section 8.

The coefficient of $\sin \bar{v}$ in (68) is three times as large as in (58), it will be noted. The difference is due to the e-term in (49), which comes in when the $\bar{v}$-term in (58) is replaced by an $\bar{M}$-term that (because it is linear with time) can then be removed from $\Delta\Omega$ and represented by $\dot{\bar{\Omega}}$, given by (67).

The quantity $\bar{n}$ that appears in (67), (69), (71) and (75) is a mean element that, like all the mean elements, is arbitrary to the extent of a constant, *viz* the constant k_n that appears in (75). Since

$$\bar{M} = \bar{M}_0 + n' t \quad ,$$

from (61), *ie* $n' = \dot{\bar{M}}$, it may seem natural just to take $\bar{n} = n'$, but alternative assumptions may be preferable and will be considered at the end of this section.

Formulae (69) to (72) are equivalent to formulae for the elements e and ω, *viz*

$$\dot{e}_1 = 0 \; , \tag{76}$$

$$e_1 = \tfrac{1}{24} \Big\{ 2 \left[12\bar{h} \cos \bar{v} + 3\bar{f} \cos (\bar{u} + \bar{\omega}) + 7\bar{f} \cos (2\bar{u} + \bar{v}) \right] + 3\bar{e} \left[4\bar{h} \cos 2\bar{v} + 10\bar{f} \cos 2\bar{u} + 3\bar{f} \cos 2(\bar{u} + \bar{v}) + k_e \right] \Big\} , \quad \ldots\ldots \tag{77}$$

$$\dot{\omega}_1 = \tfrac{1}{2}\bar{n} (4 - 5\bar{f}) \tag{78}$$

and

$$\omega_1 = \tfrac{1}{24} \Big\{ 2\bar{e}^{-1} \left[12\bar{h} \sin \bar{v} - 3\bar{f} \sin (\bar{u} + \bar{\omega}) + 7\bar{f} \sin (2\bar{u} + \bar{v}) \right] + 3 \left[4\bar{h} \sin 2\bar{v} - 2 (2 - 5\bar{f}) \sin 2\bar{u} + 3\bar{f} \sin 2(\bar{u} + \bar{v}) + k_\omega \right] \Big\} , \tag{79}$$

where

$$k_e = k_\xi \cos \bar{\omega} + k_\eta \sin \bar{\omega} \tag{80}$$

and

$$k_\omega = - k_\xi \sin \bar{\omega} + k_\eta \cos \bar{\omega} \; . \tag{81}$$

It is because of the e^{-1}-factor in (79) that the non-singular ξ_1 and η_1 should be used instead of e_1 and ω_1, but $\dot{\bar{\omega}}$ is free of this factor - in other words, $\dot{\bar{\omega}}$ only has an $O(Ke)$ effect, apart from its action in combination with n' in (73), as (69) and (71) confirm. In fact the most satisfactory way to generate (osculating) e and ω from (mean at epoch) $\bar{e}_0$ and $\bar{\omega}_0$ is by combining formulae for $\dot{\bar{e}}_1$, $\dot{\bar{\omega}}_1$, ξ_1 and η_1, such that

$$\xi = \bar{e}_0 \cos(\bar{\omega}_0 + \dot{\bar{\omega}}t) + \bar{K}\xi_1 \tag{82}$$

and

$$\eta = \bar{e}_0 \sin(\bar{\omega}_0 + \dot{\bar{\omega}}t) + \bar{K}\eta_1 \quad , \tag{83}$$

from which e and ω follow by inversion of (1) and (2).

We can now generate perturbation formulae for quantities more closely related to actual satellite position. It is convenient to express these formulae by extending the suffix-1 notation so far only used for orbital elements. We start with r and, from (40) and (37), can write

$$r = a(1 - e\cos M + e^2 \sin^2 M) + O(e^3) \quad ,$$

ie

$$r = a\left[1 - (\xi \cos U + \eta \sin U) + (\xi \sin U - \eta \cos U)^2\right] + O(e^3) \quad . \tag{84}$$

From (84) it then follows, as in section 6.1 which we are anticipating, that to $O(e)$,

$$r_1 = (\bar{r}/\bar{a})\, a_1 - \bar{a}\,(\xi_1 \cos\bar{u} + \eta_1 \sin\bar{u}) + \bar{a}\bar{e}U_1 \sin\bar{v} \quad , \tag{85}$$

a formula equivalent to one given in Ref 27, *viz*

$$\delta r = \delta a\,(1 - e\cos v) - a\,\delta e \cos v + ae\,\delta M \sin v \quad . \tag{86}$$

Now (85) reduces to

$$r_1 = \tfrac{1}{6}\bar{a}\left\{\bar{f}\cos 2\bar{u} - 6(\bar{h} - k_a) + \tfrac{1}{4}\bar{e}\left[12(3\bar{h} - 2k_a)\cos\bar{v} + 9\bar{f}\cos(\bar{u} + \omega) - 3k_\xi \cos\bar{u} - 3k_\eta \sin\bar{u}\right]\right\} \quad , \tag{87}$$

and this can be further reduced to the very simple formula

$$r_1 = \tfrac{1}{6}\bar{a}\bar{f}\cos 2\bar{u} \quad , \tag{88}$$

by making the following choices for k_a, k_ξ and k_η:

$$k_a = \bar{h} \quad , \tag{89}$$

$$k_\xi = (4\bar{h} + 3\bar{f})\cos\bar{\omega} \tag{90}$$

and

$$k_\eta = (4\bar{h} - 3\bar{f})\sin\bar{\omega} \quad . \tag{91}$$

Thus (89), (90) and (91) give 'preferred values' for these three constants and they will be referred to as such throughout this paper.

A few comments are worth adding before we pass from the perturbation in r to that in u. First, the value of $\bar{a}$ that corresponds to the choice of k_a given by (89) is often known as the 'Kozai semi-major axis', from its use in Ref 6. Second, this choice of k_a is certainly not the only useful one: comparison of (65) with (64) indicates that $\bar{a}$ can be identified, to $O(e)$, with the exact constant a', by taking $k_a = \frac{2}{3}\bar{h}$; some authors also find it convenient to take $k_a = 0$. Third, values of k_e and k_ω corresponding to the choice of k_ξ and k_η by (90) and (91) are given at once by (80) and (81), *viz*

$$k_e = 4\bar{h} + 3\bar{f} \cos 2\bar{\omega} \tag{92}$$

and

$$k_\omega = -3\bar{f} \sin 2\bar{\omega} \ . \tag{93}$$

Fourth, the constants k_ξ, k_η, k_e and k_ω are not really 'constant' at all, since $\bar{\omega}$ has a secular variation, but k_ξ and k_η are multiplied by $\bar{e}$ in (70) and (72), so their variation involves $O(Ke)$ terms which we are neglecting (see also section 9). Finally (and this ties up the last comment with an earlier remark), it is sometimes convenient (*eg* in Ref 27) to suppress the secular variation $\dot{\bar{\omega}}$, even in (82) and (83), using for $\dot{\bar{M}}$ a quantity that is really $\dot{\bar{U}}$; this involves the use of an incorrect value of $\bar{M}$, and hence of $\bar{r}$ (from the standard algorithm of section 2.5), but this can be allowed for by addition of an extra term, *viz* $-\frac{1}{2}\bar{a}\bar{e}\bar{n}t\ (4 - 5\bar{f}) \sin \bar{v}$, to (87).

We now turn to u, and from (49) can write

$$u = U + 2e \sin M + 1\tfrac{1}{4}\, e^2 \sin 2M + O(e^3) ,$$

ie

$$u = U + 2\ (\xi \sin U - \eta \cos U) + 1\tfrac{1}{4}\left[(\xi^2 - \eta^2) \sin 2U - 2\xi\eta \cos 2U\right] + O(e^3) \ . \tag{94}$$

This gives, correct to $O(e)$, again anticipating section 6.1,

$$u_1 = U_1 + 2\ (\xi_1 \sin \bar{u} - \eta_1 \cos \bar{u}) + \tfrac{1}{2}\bar{e}\left\{4U_1 \cos \bar{v} + \xi_1\left[\sin(\bar{u} + \bar{v}) + 4 \sin \bar{\omega}\right] - \eta_1\left[\cos(\bar{u} + \bar{v}) + 4 \cos \bar{\omega}\right]\right\} , \tag{95}$$

which reduces to

$$u_1 = -\tfrac{1}{12}\left\{(6 - 7\bar{f}) \sin 2\bar{u} - \bar{e}\left[6(8 - 9\bar{f}) \sin \bar{v} - (6 - \bar{f}) \sin (\bar{u} + \bar{\omega}) - 2(1 - \bar{f}) \sin (2\bar{u} + \bar{v}) + 3k_\xi \sin \bar{u} - 3k_\eta \cos \bar{u}\right]\right\} . \quad (96)$$

On substituting the preferred values of k_ξ and k_η, given by (90) and (91), we obtain

$$u_1 = -\tfrac{1}{12}\left\{(6 - 7\bar{f}) \sin 2\bar{u} - 2\bar{e}\left[6(5 - 6\bar{f}) \sin \bar{v} - (3 - 5\bar{f}) \sin (\bar{u} + \bar{\omega}) - (1 - \bar{f}) \sin (2\bar{u} + \bar{v})\right]\right\} . \quad (97)$$

If $\dot{\bar{\omega}}$ is suppressed in (82) and (83), an extra term is needed, *viz* $-\bar{e}\bar{n}t\ (4 - 5\bar{f}) \cos \bar{v}$.

We are now equipped to derive the perturbations in the cylindrical polar coordinates (r', u', c) introduced in section 2.5. From (43), (66) and (68), we have that, to $O(e)$,

$$c_1 = -\tfrac{1}{4}\bar{r} \sin 2\bar{i}\left\{(1 - k_i) \sin \bar{u} + k_\Omega \cos \bar{u} - 1\tfrac{1}{2}\bar{e}\left[2 \sin (\bar{u} + \bar{v}) - 3 \sin \bar{\omega}\right]\right\} . \quad \ldots\ldots (98)$$

If we make the preferred choice

$$k_i = 1 \quad (99)$$

and also $k_\Omega = 0$ (it is arguable that k_Ω should not have been introduced anyway, since it should naturally operate as a 'coefficient of sin 0u' whereas k_a and k_i operate as coefficients of cos 0u), (98) reduces to

$$c_1 = \tfrac{3}{8}\bar{r}\bar{e} \sin 2\bar{i}\left[2 \sin (\bar{u} + \bar{v}) - 3 \sin \bar{\omega}\right] . \quad (100)$$

Now we know from (46) that we take

$$r'_1 = r_1 , \quad (101)$$

so it remains to deal with u' . We define u'_1 on the basis that $\bar{u}'$ and $\bar{u}$ are identical; then (97) (44) and (68) give, with the preferred constants,

$$u'_1 = \tfrac{1}{12}\left\{\bar{f} \sin 2\bar{u} + 4\bar{e}\left[6\bar{h} \sin \bar{v} + \bar{f} \sin (\bar{u} + \bar{\omega})\right]\right\} , \quad (102)$$

and this could also have been obtained from the formula, correct to $O(e)$,

$$\delta u' = \delta U + \delta e\ (2 \sin v + \tfrac{1}{2}e \sin 2v) + e\ \delta M \left[2 \cos v + e\ (2 + \tfrac{1}{2} \cos 2v)\right] + \delta\Omega \cos i \quad , \qquad (103)$$

which is equivalent to the first formula on page 36 of Ref 27*.

From u_1' may be derived the quantity ℓ_1 associated with the 'L' coordinate referred to at the end of section 2.5, though we have seen that the cartesian 'RLC system' of coordinates is less satisfactory than the cylindrical 'RUC' system given by r', u' and c. Thus (47) gives

$$\ell_1 = \tfrac{1}{12} \bar{r} \left\{\bar{f} \sin 2\bar{u} + 4\bar{e} \left[6\bar{h} \sin \bar{v} + \bar{f} \sin (\bar{u} + \bar{\omega})\right]\right\} \quad , \qquad (104)$$

it being irrelevant, to first order, whether (102) is multiplied by r or $\bar{r}$; on expressing in terms of $\bar{a}$ (rather than r or $\bar{r}$), however, we get

$$\ell_1 = \tfrac{1}{24} \bar{a} \left\{2\bar{f} \sin 2\bar{u} + \bar{e} \left[48\bar{h} \sin \bar{v} + 7\bar{f} \sin (\bar{u} + \bar{\omega}) - \bar{f} \sin (2\bar{u} + \bar{v})\right]\right\} \quad .$$

(In (100), by contrast, $\bar{r}$ can be replaced by either r or $\bar{a}$.)

Preferred values have now been given for all the constants except k_n , introduced in (75). The value of k_n depends on whether or not it is regarded as essential to preserve the familiar Kepler-third-law relation, *ie* to demand that $\bar{n}^2 \bar{a}^3 = \mu$ (*cf* (4) and (62)). If we do want to preserve this relation, then it follows at once from (65) and (75) that we must set k_n equal to k_a . If we want to assign k_a and k_n independently, on the other hand, then the Kepler relation must be replaced by the more general formula

$$\bar{n}^2 \bar{a}^3 = \mu \left[1 - 3\bar{K} (k_a - k_n)\right] \quad . \qquad (105)$$

It has been remarked that "it may seem natural just to take $\bar{n} = n'$ " (since $n' = \dot{\bar{M}}$), and we have also seen that

$$\bar{a} = a' \Rightarrow k_a = \tfrac{2}{3}\bar{h} \quad ,$$

from which it follows that

$$\bar{n} = n' \Rightarrow k_n = \tfrac{2}{3}\bar{h} \quad .$$

But $\bar{h}$, *not* $\tfrac{2}{3}\bar{h}$, is the preferred value of k_a , corresponding to the Kozai semi-major axis, a_K say. This would lead to a 'mean mean motion' of no

* This follows on replacing δL in Ref 27 by $\delta U + \delta\Omega \cos i + \tfrac{1}{2}e^2\ \delta M$; the last term is O(e) and should have been included in the expression *for* δL *on* page 2 of Ref 27.

practical use, so it is common practice* to combine $k_a = \bar{h}$ with $k_n = \frac{2}{3}\bar{h}$, in which case

$$\bar{n}^2\, \bar{a}^3 \;=\; n'^2\, a_K^3 \;=\; \mu\,(1 - \overline{K}\bar{h}) \quad . \tag{106}$$

The general relation connecting $\bar{n}$, k_n and n' is

$$\bar{n} \;=\; n' + \overline{K}\bar{n}\,(1\tfrac{1}{2}\,k_n - \bar{h}) \tag{107}$$

and we will use this to derive a value for k_n that for much of this paper is not merely 'preferred' but actually essential. The associated value of $\bar{n}$ is given by

$$\bar{n} \;=\; \dot{\bar{U}} \quad (\;=\; n' + \dot{\bar{\omega}} \text{ by } (73))\ , \tag{108}$$

the significance of this value being that it permits the second-order development of section 4 to proceed on the basis that

$$2\bar{n} \int (\cos 2\bar{U},\quad \sin 2\bar{U})\, dt$$

can be replaced by ($\sin 2\bar{U}$, $-\cos 2\bar{U}$), apart from the matter of integration constants. Since $\dot{\bar{\omega}}$ is given by (78), (107) now yields

$$k_n \;=\; \tfrac{2}{3}\,(3 - 4\bar{f}) \quad , \tag{109}$$

from which (105) gives

$$\bar{n}^2\, a_K^3 \;=\; \mu\,\left[1 + \tfrac{1}{2}\overline{K}\,(6 - 7\bar{f})\right] \quad . \tag{110}$$

If, on the other hand, k_n is not limited to the value given by (109), then (73), (78) and (107) lead to the general relation

$$\dot{\bar{U}} - \bar{n} \;=\; \tfrac{1}{2}\overline{K}\bar{n}\,(6 - 8\bar{f} - 3k_n) \quad . \tag{111}$$

We may regard the right-hand side, here, as defining $\overline{K}\dot{\bar{U}}_1$, since $\dot{U}_1$ was not defined when the other $\dot{\bar{\zeta}}_1$ were introduced.

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* Kozai[6] effectively adopts a set of k's as follows: k_a and k_e are as given by (263) and (264) of section 8, such that k_a agrees with our preferred value (see section 8) to O(e) but k_e has an 'h component' that, to O(e) , is double our value given by (92); k_i is 0 (*cf* 1, here); k_Ω is 0 , as here; k_ω is exactly as given by (93) here; finally k_M (not defined till section 8) is identical with k_ω .

4 SECOND-ORDER PERTURBATIONS IN OSCULATING ELEMENTS

For any element, ζ , the secular rate of change may be expressed in the form

$$\dot{\bar{\zeta}} = \sum_{j=1}^{\infty} \bar{K}^j \dot{\bar{\zeta}}_j \tag{112}$$

and the short-periodic perturbation in the form

$$\delta\zeta = \sum_{j=1}^{\infty} \bar{K}^j \zeta_j \quad , \tag{113}$$

generalizing the notation of section 3. We are now concerned with formulae for $\dot{\bar{\zeta}}_2$ and ζ_2 , but will be neglecting $O(e)$ contributions.

The basic idea is to 'bootstrap' on the first-order solution, substituting it on the right-hand sides of the planetary equations and re-integrating. The planetary equations must be taken to $O(e)$ in this process, since

$$K\,e = K\,(\bar{e} + \delta e) \quad ,$$

where

$$\delta e = \bar{K}\,e_1 + O(\bar{K}^2) \quad ,$$

and hence Ke terms lead to $\bar{K}^2$ terms without the factor $\bar{e}$. A special procedure, free of further integration, is possible for $\zeta = a$, and this is developed first. It may be compared with a re-derivation of a_2 by the general procedure.

4.1 Perturbation in a (special method)

From (63) and (11) it follows that

$$a = a' + K\,a'\,q^{-2}\,(p/r)^3\,(f \cos 2u + \tfrac{2}{3}h) \quad , \tag{114}$$

this being an exact result. Recovery of the first-order a_1 , given by (65) to $O(e)$, is immediate on expanding p/r by (12), on the basis that

$$a' = \bar{a}\,(1 + k_a - \tfrac{2}{3}\bar{h}) \quad . \tag{115}$$

If we are to be able to write

$$a = a' + \bar{K}\,\bar{a}\,\bar{q}^{-2}\,(\bar{p}/\bar{r})^3\,(\bar{f} \cos 2\bar{u} + \tfrac{2}{3}\bar{h}) + \bar{K}^2 a_2 \quad ,$$

there will be five sources of terms in a_2 : (i) the 'a variation', lost on replacing K by $\bar{K}$; (ii) the generalization from a' to $\bar{a}$, these being connected by (115); (iii) the perturbation in $(p/r)^3$; (iv) the variation in i , which affects f and h ; and (v) the perturbation in u . The contributions from these sources are as follows, neglecting terms that are $O(e)$ in a_2 :-

(i) $\quad - 2\bar{a}\,(\bar{f}\cos 2\bar{u} + \tfrac{2}{3}\bar{h})(\bar{f}\cos 2\bar{u} + k_a)$;

(ii) $\quad \bar{a}\,(\bar{f}\cos 2\bar{u} + \tfrac{2}{3}\bar{h})(k_a - \tfrac{2}{3}\bar{h})$;

(iii) $\quad 3\bar{a}\,(\bar{f}\cos 2\bar{u} + \tfrac{2}{3}\bar{h})(\bar{h} + \tfrac{5}{6}\bar{f}\cos 2\bar{u})$;

(iv) $\quad \bar{a}\bar{f}\,(1 - \bar{f})(\cos 2\bar{u} - 1)(\cos 2\bar{u} + k_i)$;

(v) $\quad \tfrac{1}{6}\bar{a}\bar{f}\,(6 - 7\bar{f})\sin^2 2\bar{u}$.

Summation of the contributions gives

$$a_2 = \tfrac{1}{3}\bar{a}\left\{\bar{f}^2\cos 4\bar{u} + \bar{f}\left[5 - 9\bar{f} - 3k_a + 3k_i(1 - \bar{f})\right]\cos 2\bar{u} + \tfrac{1}{3}\left[14 - 33\bar{f} + 24\bar{f}^2 - 6k_a\bar{h} - 9k_i\bar{f}(1 - \bar{f})\right]\right\} . \qquad (116)$$

The constant term here is of no great general importance, and in the next section an arbitrary constant k_{2a} is introduced, by equation (121), in analogy with k_a. Comparison of (116) and (121) enables us to obtain the expression for k_{2a} appropriate to an interpretation of $\bar{a}$ as a' , *ie* to obtain the second-order relation of a to a' . We set k_a to $\tfrac{2}{3}\bar{h}$, therefore, and obtain

$$k_{2a} = \tfrac{1}{9}\left[(10 - 21\bar{f} + 15\bar{f}^2) - 9k_i\bar{f}(1 - \bar{f})\right] . \qquad (117)$$

An application for this result is found in section 9.7.

4.2 Perturbation in a (general method)

The starting point is the exact equation (23). From this it follows, using (12), (51) and (52), that to $O(e)$

$$\dot{a} = -\tfrac{1}{2}Kna\left\{4f\sin 2U + e\left[4h\sin v - f\sin(u + \omega) + 21f\sin(2u + v)\right]\right\} . \qquad \ldots\ldots (118)$$

The object of replacing $\sin 2u$ by $\sin 2U$ in the main term is to permit first-order integration with respect to time (as opposed to changing the integration variable to true anomaly, by invoking (54)). The *first-order* solution then follows at once in the form

$$\delta a \;=\; \tfrac{1}{2}\bar{K}\bar{a}\left\{2\bar{f}\cos 2\bar{U} + \bar{e}\left[4\bar{h}\cos\bar{v} - \bar{f}\cos(\bar{u}+\bar{\omega}) + 7\bar{f}\cos(2\bar{u}+\bar{v})\right]\right\} \;, \qquad \ldots\ldots (119)$$

which is equivalent to (65), with $k_a = 0$, if allowance is made, by (53), for the difference between $\cos 2\bar{U}$ and $\cos 2\bar{u}$.

It is now possible to expand $\dot{a}$, as given by (118), in terms of the first-order reference solutions for a, e, i, ω, U and n. Thus there are six contributions to the total $\dot{a}$ which are second-order and they may be denoted by $\Delta_a\dot{a}$, $\Delta_e\dot{a}$ *etc*, where

$$\Delta_\zeta\dot{a} \;=\; \frac{\partial\dot{a}}{\partial\zeta}\,\Delta\zeta \;.$$

Remembering that K is a function of a, and that we are neglecting $O(K^2e)$ perturbations, we obtain

$$\Delta_a\dot{a} \;=\; -\,K\dot{a}\,(f\cos 2u + k_a) \;,$$

$$\Delta_e\dot{a} \;=\; -\,\tfrac{1}{24}K^2na\left[4h\sin v - f\sin(u+\omega) + 21f\sin(2u+v)\right]\times$$
$$\times\left[12h\cos v + 3f\cos(u+\omega) + 7f\cos(2u+v)\right] \;,$$

$$\Delta_i\dot{a} \;=\; K\dot{a}\,(1-f)(\cos 2u + k_i) \;,$$

$$\Delta_\omega\dot{a} \;=\; \tfrac{1}{24}K^2na\left[4h\cos v + f\cos(u+\omega) + 21f\cos(2u+v)\right]\times$$
$$\times\left[12h\sin v - 3f\sin(u+\omega) + 7f\sin(2u+v)\right] \;,$$

$$\Delta_U\dot{a} \;=\; -\,\tfrac{1}{2}K\dot{a}\,(2-5f)\cos 2u$$

and

$$\Delta_n\dot{a} \;=\; -\,1\tfrac{1}{2}K\dot{a}\,(f\cos 2u + k_n) \;.$$

It is immaterial whether the quantities on the right-hand sides are regarded as osculating or mean, so 'bars' are omitted for convenience. The value of k_n must be taken from (109), however.

It now follows, after some tedious reduction, that the total second-order contribution to $\dot{a}$ is given by

$$\Delta\dot{a} \;=\; -\,\tfrac{3}{4}K^2naf\left\{2f\sin 4u + \left[5 - 9f - 3k_a + 3k_i(1-f)\right]\sin 2u\right\} \;. \qquad (120)$$

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The formula for second-order δa follows immediately, and we may write

$$a_2 = \tfrac{1}{3}\bar{a}\left\{\bar{f}^2 \cos 4\bar{u} + \bar{f}\left[5 - 9\bar{f} - 3k_a + 3k_i(1 - \bar{f})\right] \cos 2\bar{u} + 3k_{2a}\right\} , \qquad \text{......(121)}$$

where it has been convenient to introduce a second-order constant, k_{2a}, akin to k_a. Clearly (121) is in agreement with (116).

Taking the preferred values of $\bar{h}$ and 1 for k_a and k_i respectively, we get

$$a_2 = \tfrac{1}{3}\bar{a}\,(\bar{f}^2 \cos 4\bar{u} + 5\bar{f}\bar{h} \cos 2\bar{u} + 3k_{2a}) , \qquad (122)$$

where the preferred value of k_{2a}, as we shall see in section 6.1, is given by

$$k_{2a} = \bar{h}^2 - \tfrac{1}{72}\bar{f}^2 , \qquad (123)$$

ie by

$$k_{2a} = 1 - 3\bar{f} + 2\tfrac{17}{72}\bar{f}^2 .$$

4.3 Perturbation in i

The starting point is the exact equation (25), which to $O(e)$ gives

$$\dot{i} = -\tfrac{1}{4}Kn \sin 2i \left\{2 \sin 2U - e\left[\sin(u + \omega) - 7 \sin(2u + v)\right]\right\} . \qquad (124)$$

The first-order solution follows at once, in the form

$$\delta i = \tfrac{1}{12}\bar{K} \sin 2\bar{i} \left\{3 \cos 2\bar{U} - \bar{e}\left[3 \cos(\bar{u} + \bar{\omega}) - 7 \cos(2\bar{u} + \bar{v})\right]\right\} , \qquad (125)$$

equivalent to (66) (with $k_i = 0$).

We expand $\dot{i}$, as given by (124), in terms of the first-order reference solutions for a, e *etc*, just as in section 4.2, obtaining six second-order contributions to $\Delta\dot{i}$, *viz*

$$\Delta_a \dot{i} = -2K\dot{i}\,(f \cos 2u + k_a) \quad ,$$

$$\Delta_e \dot{i} = \tfrac{1}{48}K^2 n \sin 2i \left[\sin (u + \omega) - 7 \sin (2u + v)\right] \times$$
$$\times \left[12h \cos v + 3f \cos (u + \omega) + 7f \cos (2u + v)\right] \quad ,$$

$$\Delta_i \dot{i} = \tfrac{1}{2}K\dot{i} \cos 2i \,(\cos 2u + k_i) \quad ,$$

$$\Delta_\omega \dot{i} = \tfrac{1}{48}K^2 n \sin 2i \left[\cos (u + \omega) + 7 \cos (2u + v)\right] \times$$
$$\times \left[12h \sin v - 3f \sin (u + \omega) + 7f \sin (2u + v)\right] \quad ,$$

$$\Delta_U \dot{i} = -\tfrac{1}{2}K\dot{i}\,(2 - 5f) \cos 2u$$

and

$$\Delta_n \dot{i} = -1\tfrac{1}{2}K\dot{i}\,(f \cos 2u + k_n) \quad .$$

Tedious reduction, as for $\Delta\dot{a}$, leads to the total Δi given by

$$\Delta\dot{i} = \tfrac{1}{24}K^2 n \sin 2i \left\{(3 + 5f) \sin 4u + 6 \left[f + 4k_a - k_i(1 - 2f)\right] \sin 2u\right\} \quad , \qquad \ldots\ldots (126)$$

from which the formula for second-order δi follows immediately; thus

$$i_2 = -\tfrac{1}{96} \sin 2\bar{i} \left\{(3 + 5\bar{f}) \cos 4\bar{u} + 12 \left[\bar{f} + 4k_a - k_i(1 - 2\bar{f})\right] \cos 2\bar{u} + k_{2i}\right\} \quad , \qquad \ldots\ldots (127)$$

with a convenient second-order constant introduced.

With the preferred values of k_a and k_i , we get

$$i_2 = -\tfrac{1}{96} \sin 2\bar{i} \left[(3 + 5\bar{f}) \cos 4\bar{u} + 36(1 - \bar{f}) \cos 2\bar{u} + k_{2i}\right] \quad , \qquad (128)$$

where the preferred value of k_{2i} is given (as we shall see in section 6.3) by

$$k_{2i} = 33 - 49\bar{f} \quad . \qquad (129)$$

4.4 Perturbation in Ω

The exact starting equation is (26), which to $O(e)$ gives

$$\dot{\Omega} = -\tfrac{1}{2}Kn \cos i \left\{2\,(1 - \cos 2U) + e\left[6 \cos v + \cos (u + \omega) - 7 \cos (2u + v)\right]\right\} \quad . \qquad \ldots\ldots (130)$$

The first-order solution follows at once, and is composed of a secular term, with $\dot{\bar{\Omega}}_1$ given by (67), and a periodic term in the form

$$\delta\Omega = \tfrac{1}{6}\bar{K} \cos \bar{i} \left\{3 \sin 2\bar{U} - \bar{e} \left[18 \sin \bar{v} + 3 \sin (\bar{u} + \bar{\omega}) - 7 \sin (2\bar{u} + \bar{v})\right]\right\} , \qquad \text{...... (131)}$$

equivalent to (68) (with $k_\Omega = 0$).

On expanding $\dot{\Omega}$ relative to the first-order reference solutions for a, e *etc*, we obtain the following six second-order contributions to $\Delta\dot{\Omega}$:

$$\Delta_a\dot{\Omega} = -2K\dot{\Omega} (f \cos 2u + k_a) ,$$

$$\Delta_e\dot{\Omega} = -\tfrac{1}{24}K^2 n \cos i \left[6 \cos v + \cos (u + \omega) - 7 \cos (2u + v)\right] \times$$
$$\times \left[12h \cos v + 3f \cos (u + \omega) + 7f \cos (2u + v)\right] ,$$

$$\Delta_i\dot{\Omega} = -\tfrac{1}{2}K\dot{\Omega}f (\cos 2u + k_i) ,$$

$$\Delta_\omega\dot{\Omega} = -\tfrac{1}{24}K^2 n \cos i \left[6 \sin v - \sin (u + \omega) - 7 \sin (2u + v)\right] \times$$
$$\times \left[12h \sin v - 3f \sin (u + \omega) + 7f \sin (2u + v)\right] ,$$

$$\Delta_U\dot{\Omega} = -K\dot{\Omega} (2 - 5f) \cos^2 u$$

and

$$\Delta_n\dot{\Omega} = -1\tfrac{1}{2}K\dot{\Omega} (f \cos 2u + k_n) .$$

The total $\Delta\dot{\Omega}$ is given by

$$\Delta\dot{\Omega} = -\tfrac{1}{6}K^2 n \cos i \left\{(3 + f) \cos 4u - 3(2f - 4k_a - k_i f) \cos 2u + (15 - 19f - 12k_a - 3k_i f - 9k_n)\right\} ; \qquad \text{...... (132)}$$

thus there is a secular component of the second-order $\Delta\Omega$, and k_n has been left in (132) because the value required depends on the chosen $\bar{n}$, which appears explicitly in $\dot{\bar{\Omega}}$. We get

$$\dot{\bar{\Omega}}_2 = -\tfrac{1}{6}\bar{n} \cos \bar{i} (15 - 19\bar{f} - 12k_a - 3k_i\bar{f} - 9k_n) \qquad (133)$$

and

$$\Omega_2 = -\tfrac{1}{24} \cos \bar{i} \left[(3 + \bar{f}) \sin 4\bar{u} - 6(2\bar{f} - 4k_a - k_i\bar{f}) \sin 2\bar{u}\right] , \qquad (134)$$

no $k_{2\Omega}$ constant being required (it would be the coefficient of $\sin 0u$!) .

With the preferred values of k_a and k_i, we get

$$\dot{\Omega}_2 = -\tfrac{1}{6}\bar{n} \cos \bar{i} \, (3 - 4f - 9k_n) \qquad (135)$$

and

$$\Omega_2 = -\tfrac{1}{24} \cos \bar{i} \left[(3 + \bar{f}) \sin 4\bar{u} + 6(4 - 7\bar{f}) \sin 2\bar{u}\right] . \qquad (136)$$

With our usual value of k_n, we get (*cf* (271) in section 9.5)

$$\dot{\Omega}_2 = \tfrac{5}{6}\, \bar{n} \cos \bar{i} \, (3 - 4\bar{f}) . \qquad (137)$$

4.5 Perturbation in ξ

The exact starting equation is (31), which to $O(e)$ gives

$$\begin{aligned}\dot{\xi} = -\tfrac{1}{4}Kn \Big\{&(4h + f) \sin U + 7f \sin 3U \\ &+ 2e \big[(4 - 5f) \sin \omega + (7 - 8f) \sin (u + v) \\ &- (1 + 2f) \sin (2u + \omega) + 17f \sin (3u + v)\big]\Big\} .\end{aligned}$$

...... (138)

The first-order solution follows at once, in a form consisting of a secular term, with $\dot{\xi}_1$ given by (69), and a periodic term

$$\begin{aligned}\delta\xi = \tfrac{1}{24}\bar{K} \Big\{&6(4\bar{h} + \bar{f}) \cos \bar{U} + 14\bar{f} \cos 3\bar{U} + 3\bar{e} \times \\ &\times \big[2(7 - 8\bar{f}) \cos (\bar{u} + \bar{v}) - 2(1 + 2\bar{f}) \cos (2\bar{u} + \bar{\omega}) + 17\bar{f} \cos (3\bar{u} + \bar{v})\big]\Big\} ,\end{aligned}$$

...... (139)

equivalent to (70) with k_ξ taken as* $2(4\bar{h} + \bar{f}) \cos \bar{\omega}$.

On expanding $\dot{\xi}$ relative to the usual first-order reference solutions, we obtain:

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* This k_ξ and the corresponding k_η (see section 4.6) are equivalent to the (effective) k_e and k_ω of Ref 25 that we encounter in section 5.2.

$$\Delta_a \dot{\xi} = -2K\dot{\xi}\ (f \cos 2u + k_a) \ ,$$

$$\Delta_e \dot{\xi} = -\tfrac{1}{24}K^2 n \left[(4 - 5f) \sin \omega + (7 - 8f) \sin (u + v) - (1 + 2f) \sin (2u + \omega) + 17f \sin (3u + v)\right]\left[12h \cos v + 3f \cos (u + \omega) + 7f \cos (2u + v)\right] \ ,$$

$$\Delta_i \dot{\xi} = \tfrac{1}{4}K^2 nf\ (1 - f)(5 \sin u - 7 \sin 3u)(\cos 2u + k_i) \ ,$$

$$\Delta_\omega \dot{\xi} = -\tfrac{1}{24}K^2 n \left[(4 - 5f) \cos \omega - (7 - 8f) \cos (u + v) - (1 + 2f) \cos (2u + \omega) - 17f \cos (3u + v)\right]\left[12h \sin v - 3f \sin (u + \omega) + 7f \sin (2u + v)\right] \ ,$$

$$\Delta_U \dot{\xi} = \tfrac{1}{16}K^2 n\ (2 - 5f)\ \left[(4h + f) \cos u + 21f \cos 3u\right] \sin 2u$$

and

$$\Delta_n \dot{\xi} = -1\tfrac{1}{2}K\dot{\xi}\ (f \cos 2u + k_n) \ .$$

The total $\Delta\dot{\xi}$ is given by

$$\Delta\dot{\xi} = \tfrac{1}{96}K^2 n \Big\{5f\ (14 - 17f) \sin 5u + \left[72 - 346f + 449f^2 + 336fk_a - 168k_i f(1 - f)\right] \times \sin 3u - 2\left[108 - 200f + 77f^2 - 24k_a(4 - 5f) - 60k_i f(1 - f)\right] \sin u\Big\} \ , \qquad \text{...... (140)}$$

whence

$$\xi_2 = -\tfrac{1}{288}\Big\{3\bar{f}\ (14 - 17\bar{f}) \cos 5\bar{u} + \left[72 - 346\bar{f} + 449\bar{f}^2 + 336\bar{f}k_a - 168k_i\bar{f}(1 - \bar{f})\right] \cos 3\bar{u} - 6\left[108 - 200\bar{f} + 77\bar{f}^2 - 24k_a(4 - 5\bar{f}) - 60k_i\bar{f}(1 - \bar{f})\right] \cos \bar{u}\Big\} \ , \qquad \text{...... (141)}$$

no constant being introduced.

With the preferred values of k_a and k_i, we get

$$\xi_2 = -\tfrac{1}{288}\Big\{3\bar{f}\ (14 - 17\bar{f}) \cos 5\bar{u} + (72 - 178\bar{f} + 113\bar{f}^2) \cos 3\bar{u} - 6(12 + 4\bar{f} - 43\bar{f}^2) \cos \bar{u}\Big\} \ . \qquad (142)$$

4.6 Perturbation in η

The exact starting equation is (32), which to $O(e)$ gives

$$\dot{\eta} = \tfrac{1}{4}Kn \Big\{(4h - f) \cos U + 7f \cos 3U + 2e\left[(4 - 5f) \cos \omega + 5(1 - 2f) \cos (u + v) - (1 + 2f) \cos (2u + \omega) + 17f \cos (3u + v)\right]\Big\} \ . \qquad \text{...... (143)}$$

The first-order solution follows at once, in a form consisting of a secular term, with $\bar{\dot{\eta}}_1$ given by (71), and a periodic term

$$\delta\eta = \tfrac{1}{24}K\Big\{6(4\bar{h} - \bar{f})\sin\bar{U} + 14\bar{f}\sin 3\bar{U} + 3\bar{e}\times$$
$$\times\big[10(1-2\bar{f})\sin(\bar{u}+\bar{v}) - 2(1+2\bar{f})\sin(2\bar{u}+\bar{\omega}) + 17\bar{f}\sin(3\bar{u}+\bar{v})\big]\Big\}, \quad \ldots\ldots (144)$$

equivalent to (72) with k_η taken as $2(4\bar{h} - \bar{f})\sin\bar{\omega}$.

On expanding $\dot{\eta}$ relative to the usual first-order reference solutions, we obtain:

$$\Delta_a\dot{\eta} = -2K\dot{\eta}\,(f\cos 2u + k_a) \ ,$$

$$\Delta_e\dot{\eta} = \tfrac{1}{24}K^2 n\,\big[(4-5f)\cos\omega + 5(1-2f)\cos(u+v) - (1+2f)\cos(2u+\omega)$$
$$+ 17f\cos(3u+v)\big]\big[12h\cos v + 3f\cos(u+\omega) + 7f\cos(2u+v)\big] \ ,$$

$$\Delta_i\dot{\eta} = -1\tfrac{3}{4}K^2 nf\,(1-f)\,(\cos u - \cos 3u)(\cos 2u + k_i) \ ,$$

$$\Delta_\omega\dot{\eta} = -\tfrac{1}{24}K^2 n\,\big[(4-5f)\sin\omega - 5(1-2f)\sin(u+v) - (1+2f)\sin(2u+\omega)$$
$$- 17f\sin(3u+v)\big]\big[12h\sin v - 3f\sin(u+\omega) + 7f\sin(2u+v)\big] \ ,$$

$$\Delta_U\dot{\eta} = \tfrac{1}{16}K^2 n\,(2-5f)\big[(4h-f)\sin u + 21f\sin 3u\big]\sin 2u$$

and

$$\Delta_n\dot{\eta} = -1\tfrac{1}{2}K\dot{\eta}\,(f\cos 2u + k_n) \ .$$

The total $\Delta\dot{\eta}$ is given by

$$\Delta\dot{\eta} = -\tfrac{1}{96}K^2 n\,\Big\{5f\,(14-17f)\cos 5u + \big[72 - 310f + 395f^2 + 336fk_a$$
$$- 168k_i f(1-f)\big]\cos 3u - 2\big[84 - 224f + 155f^2$$
$$- 24k_a(4-7f) - 84k_i f(1-f)\big]\cos u\Big\} \ , \qquad (145)$$

whence

$$\eta_2 = -\tfrac{1}{288}\Big\{3\bar{f}\,(14-17\bar{f})\sin 5\bar{u} + \big[72 - 310\bar{f} + 395\bar{f}^2 + 336\bar{f}k_a$$
$$- 168k_i\bar{f}(1-\bar{f})\big]\sin 3\bar{u} - 6\big[84 - 224\bar{f} + 155\bar{f}^2$$
$$- 24k_a(4-7\bar{f}) - 84k_i\bar{f}(1-\bar{f})\big]\sin\bar{u}\Big\} \ , \qquad (146)$$

no constant being introduced.

With the preferred values of k_a and k_i , we get

$$\eta_2 = -\tfrac{1}{288}\Big\{3\bar{f}\ (14 - 17\bar{f})\ \sin 5\bar{u} + (72 - 142\bar{f} + 59\bar{f}^2)\ \sin 3\bar{u} + 6(12 - 4\bar{f} + 13\bar{f}^2)\ \sin \bar{u}\Big\}\ . \qquad (147)$$

4.7 <u>Perturbation in $(\sigma + \omega)$</u>

From the exact equations (27) and (28) it follows that, to $O(e)$,

$$\dot{\sigma} + \dot{\omega} = \tfrac{1}{8}Kn\Big\{8\big[(3 - 4f) - (1 - 4f)\cos 2U\big] + e\big[2(38 - 51f)\cos v + (4 - 17f)\cos(u + \omega) - 7(4 - 17f)\cos(2u + v)\big]\Big\}\ . \qquad \ldots\ldots (148)$$

The first-order solution follows at once; it consists of a secular term, given by

$$\dot{\bar{\sigma}}_1 + \dot{\bar{\omega}}_1 = \bar{n}\ (3 - 4\bar{f})\ , \qquad (149)$$

and a periodic term in the form

$$\delta\sigma + \delta\omega = -\tfrac{1}{24}\bar{K}\Big\{12\ (1 - 4\bar{f})\ \sin 2\bar{U} - \bar{e}\ \big[6(38 - 51\bar{f})\ \sin \bar{v} + 3(4 - 17\bar{f})\ \sin(\bar{u} + \bar{\omega}) - 7(4 - 17\bar{f})\ \sin(2\bar{u} + \bar{v})\big]\Big\}\ . \qquad (150)$$

This is equivalent, by (53), to a result that could be added to equations (65) to (75), *viz*

$$\sigma_1 + \omega_1 = -\tfrac{1}{48}\Big\{24\ (1 - 4\bar{f})\ \sin 2\bar{u} - \bar{e}\ \big[6(38 - 51\bar{f})\ \sin \bar{v} - 3(4 - 15\bar{f})\ \sin(\bar{u} + \bar{\omega}) - (4 - 23\bar{f})\ \sin(2\bar{u} + \bar{v})\big]\Big\}\ . \qquad (151)$$

On expanding $(\dot{\sigma} + \dot{\omega})$ relative to the first-order reference solutions as usual, we obtain:

$$\Delta_a(\dot{\sigma} + \dot{\omega}) = -\ 2K\ (\dot{\sigma} + \dot{\omega})(f\cos 2u + k_a)\ ,$$

$$\Delta_e(\dot{\sigma} + \dot{\omega}) = \tfrac{1}{96}K^2 n\ \big[2(38 - 51f)\cos v + (4 - 17f)\cos(u + \omega) - 7(4 - 17f)\cos(2u + v)\big]\big[12h\cos v + 3f\cos(u + \omega) + 7f\cos(2u + v)\big]\ ,$$

$$\Delta_i(\dot{\sigma} + \dot{\omega}) = -\ 4K^2 nf\ (1 - f)(1 - \cos 2u)(\cos 2u + k_i)\ ,$$

$$\Delta_\omega(\dot\sigma + \dot\omega) = \tfrac{1}{96}K^2 n \left[2(38 - 51f)\sin v - (4 - 17f)\sin(u + \omega) - 7(4 - 17f)\sin(2u + v)\right]\left[12h \sin v - 3f \sin(u + \omega) + 7f \sin(2u + v)\right] ,$$

$$\Delta_U(\dot\sigma + \dot\omega) = -\tfrac{1}{2}K^2 n\,(1 - 4f)(2 - 5f)\sin^2 2u$$

and

$$\Delta_n(\dot\sigma + \dot\omega) = -1\tfrac{1}{2}K\,(\dot\sigma + \dot\omega)(f \cos 2u + k_n) .$$

The total $\Delta(\dot\sigma + \dot\omega)$ is given by

$$\Delta(\dot\sigma + \dot\omega) = \tfrac{1}{48}K^2 n \Big\{(24 - 4f - 73f^2)\cos 4u - 4\left[f(64 - 51f) - 24k_a(1 - 4f) - 48k_i f(1 - f)\right]\cos 2u + \left[(432 - 1052f + 637f^2) - 24(3 - 4f)(4k_a + 3k_n) - 192k_i f(1 - f)\right]\Big\} , \qquad \ldots\ldots (152)$$

where a term in k_n has been retained in the secular component for the same reason as with $\Delta\dot\Omega$. We get

$$(\dot{\bar\sigma} + \dot{\bar\omega})_2 = \tfrac{1}{48}\bar n \left[(432 - 1052\bar f + 637\bar f^2) - 24(3 - 4\bar f)(4k_a + 3k_n) - 192k_i \bar f(1 - \bar f)\right] \qquad \ldots\ldots (153)$$

and

$$(\sigma + \omega)_2 = \tfrac{1}{192}\Big\{(24 - 4\bar f - 73\bar f^2)\sin 4\bar u - 8\left[\bar f(64 - 51\bar f) - 24k_a(1 - 4\bar f) - 48k_i \bar f(1 - \bar f)\right]\sin 2\bar u\Big\} . \qquad \ldots\ldots (154)$$

With the preferred values of k_a and k_i , we get

$$(\dot{\bar\sigma} + \dot{\bar\omega})_2 = \tfrac{1}{48}\bar n \left[(144 - 428\bar f + 253\bar f^2) - 72k_n(3 - 4\bar f)\right] \qquad (155)$$

and

$$(\sigma + \omega)_2 = \tfrac{1}{192}\Big\{(24 - 4\bar f - 73\bar f^2)\sin 4\bar u + 8(24 - 148\bar f + 147\bar f^2)\sin 2\bar u\Big\} , \qquad \ldots\ldots (156)$$

and if we use our usual value of k_n we get

$$(\dot{\bar\sigma} + \dot{\bar\omega})_2 = -\tfrac{1}{48}\bar n\,(288 - 724\bar f + 515\bar f^2) . \qquad (157)$$

4.8 Perturbation in n and in $\int_0^t n\,dt$

To be able to analyse the perturbation in U, as we shall in section 4.9, we need a formula for the perturbation in $\int_0^t n\,dt$, to be able to combine with $\Delta(\sigma + \omega)$, following (20) and (3). Thus we first need the formula for n_2 that corresponds to the perturbation in n.

The perturbation in n could be obtained by the usual method, involving integration, but it is more direct to obtain it from the perturbation in a, making use of the fundamental Kepler relation (4). This is possible even though, because of choosing independent k-constants for n and a, $\bar{n}$ and $\bar{a}$ will not in general satisfy the relation themselves. We introduce $\hat{\mu}_1$ and $\hat{\mu}_2$, therefore, dependent only on $\bar{f}$ (so long as we work only to first order in e), such that

$$\bar{n}^2\,\bar{a}^3 = \mu\,(1 + \bar{K}\,\hat{\mu}_1 + \bar{K}^2\,\hat{\mu}_2) \quad . \tag{158}$$

This is equivalent to the relation

$$n^2\,a^3 = \bar{n}^2\,\bar{a}^3\,(1 + \bar{K}\,\hat{\mu}_1 + \bar{K}^2\,\hat{\mu}_2)^{-1} \quad , \tag{159}$$

where a strong reason for introducing the relation in this form, rather than reciprocally, is that the final expression for $\hat{\mu}_2$ - see (185) - becomes very simple; reciprocally, the relation is

$$n^2\,a^3 = \bar{n}^2\,\bar{a}^3\,[1 - \bar{K}\,\hat{\mu}_1 - \bar{K}^2\,(\hat{\mu}_2 - \hat{\mu}_1^2)] \quad . \tag{160}$$

The general first-order result, previously given by (105), may now be written in the form

$$\hat{\mu}_1 = 3\,(k_n - k_a) \quad , \tag{161}$$

which with the preferred k_a and k_n reduces (*cf* (110)) to

$$\hat{\mu}_1 = \tfrac{1}{2}\,(6 - 7\bar{f}) \quad . \tag{162}$$

But

$$a = \bar{a} + \bar{K}\,a_1 + \bar{K}^2\,a_2$$

and

$$n = \bar{n} + \bar{K}\,n_1 + \bar{K}^2\,n_2 \, ,$$

from which it follows that

$$n^2 a^3 = \bar{n}^2 \bar{a}^3 (1 + \bar{K}\hat{a}_1 + \bar{K}^2\hat{a}_2)^3 (1 + \bar{K}\hat{n}_1 + \bar{K}^2\hat{n}_2)^2 \quad , \tag{163}$$

on defining $\hat{a}_1$ to be $a_1/\bar{a}$ *etc* for convenience. On comparing (160) and (163) it follows that

$$\hat{\mu}_1 = -(3\hat{a}_1 + 2\hat{n}_1) \tag{164}$$

and

$$\hat{\mu}_2 = 6\hat{a}_1^2 + 6\hat{a}_1\hat{n}_1 + 3\hat{n}_1^2 - 3\hat{a}_2 - 2\hat{n}_2 \quad . \tag{165}$$

There is nothing new in (164), which just leads to (161) again, but (165) leads to the desired formula for n_2 . To see this, we start with formulae for $\hat{a}_1$, $\hat{n}_1$ and $\hat{a}_2$, which follow from (65), (75) and (121) respectively, whereupon (165) gives

$$2\hat{n}_2 + \hat{\mu}_2 = \tfrac{1}{8}\Big\{7\bar{f}^2 \cos 4\bar{u} - 4\bar{f}\,\big[2(5 - 9\bar{f}) - 12k_a + 6k_i(1 - \bar{f}) - 9k_n\big] \cos 2\bar{u}$$
$$+ 3\,(5\bar{f}^2 + 16k_a^2 - 24k_a k_n + 18k_n^2 - 8k_{2a})\Big\} \quad . \tag{166}$$

(It is legitimate to leave k_n in the coefficient of $\cos 2\bar{u}$, in spite of the remarks of section 3, because it arises through expansion of (165) and not from integration.)

But $\hat{\mu}_2$ must be independent of $\bar{u}$, so we must in consequence of (166) be able to write

$$n_2 = \tfrac{1}{16}\bar{n}\Big\{7\bar{f}^2 \cos 4\bar{u} - 4\bar{f}\,\big[2(5 - 9\bar{f}) - 12k_a + 6k_i(1 - \bar{f}) - 9k_n\big] \cos 2\bar{u} + 16k_{2n}\Big\} ,$$
$$\ldots\ldots \tag{167}$$

where the introduced constant k_{2n} is such that

$$\hat{\mu}_2 = \tfrac{3}{8}\,(5\bar{f}^2 + 16k_a^2 - 24k_a k_n + 18k_n^2 - 8k_{2a}) - 2k_{2n} \quad . \tag{168}$$

With the preferred values of k_a, k_i and k_n , (167) reduces to

$$n_2 = \tfrac{1}{16}\bar{n}\Big\{7\bar{f}^2 \cos 4\bar{u} + 8\bar{f}\,(7 - 9\bar{f}) \cos 2\bar{u} + 16k_{2n}\Big\} \tag{169}$$

and (168) to

$$\hat{\mu}_2 = \tfrac{3}{8}\,(40 - 104\bar{f} + 73\bar{f}^2 - 8k_{2a}) - 2k_{2n} \quad . \tag{170}$$

To obtain the perturbation in $\int_0^t n\,dt$, we observe that

$$n = \bar{n} + \bar{K}\, n_1 + \bar{K}^2\, n_2 \quad , \tag{171}$$

where n_1 - *cf* (75) and (119) - is given by

$$n_1 = -1\tfrac{1}{2}\bar{n}\left\{\bar{f}\cos 2\bar{U} + k_n + \tfrac{1}{2}\bar{e}\left[4\bar{h}\cos\bar{v} - \bar{f}\cos(\bar{u}+\bar{\omega}) + 7\bar{f}\cos(2\bar{u}+\bar{v})\right]\right\} . \tag{172}$$

Then writing the required integral as a sum of secular and periodic components, we have

$$\int_0^t n\,dt = (\bar{n} + \Delta\bar{n})\,t + \bar{K}\int_1 + \bar{K}^2\int_2 , \tag{173}$$

where (172) and (169) imply that

$$\Delta\bar{n}/\bar{n} = -1\tfrac{1}{2}\,\bar{K}\,k_n + \bar{K}^2\,k_{2n} \quad . \tag{174}$$

For $\int_1$, we integrate the periodic terms of (172), obtaining

$$\int_1 = -\tfrac{1}{4}\left\{3\bar{f}\sin 2\bar{U} + \bar{e}\left[12\bar{h}\sin\bar{v} - 3\bar{f}\sin(\bar{u}+\bar{\omega}) + 7\bar{f}\sin(2\bar{u}+\bar{v})\right]\right\} . \tag{175}$$

For $\int_2$, similarly, we integrate (167) to obtain (since k_n must necessarily be given its preferred value),

$$\int_2 = \tfrac{1}{64}\left\{7\bar{f}^2\sin 4\bar{u} + 16\bar{f}\left[4 - 3\bar{f} + 6k_a - 3k_i(1-\bar{f})\right]\sin 2\bar{u}\right\} , \tag{176}$$

which - or from (169) directly - with the preferred values of k_a and k_i as well, reduces to

$$\int_2 = \tfrac{1}{64}\left\{7\bar{f}^2\sin 4\bar{u} + 16\bar{f}\,(7 - 9\bar{f})\sin 2\bar{u}\right\} . \tag{177}$$

4.9 Perturbation in U

From (20 and (3) we obtain at once, on combining (149) with (174) and (150) with (175), the first-order secular result for $\dot{\bar{U}}_1$, as given by (111), and the periodic term given by

$$U_1 = -\frac{1}{24}\left\{6(2 - 5\bar{f}) \sin 2\bar{U} - \bar{e}\left[6(26 - 33\bar{f}) \sin \bar{v} + 3(4 - 11\bar{f}) \sin (\bar{u} + \bar{\omega}) - 7(4 - 11\bar{f}) \sin (2\bar{u} + \bar{v})\right]\right\} , \quad (178)$$

which is equivalent to (74).

The expression for U_2 is given by combining (154) and (176); thus

$$U_2 = \frac{1}{48}\left\{(6 - \bar{f} - 13\bar{f}^2) \sin 4\bar{u} - 2\left[\bar{f}(40 - 33\bar{f}) - 12k_a(2 - 5\bar{f}) - 30k_i\bar{f}(1 - \bar{f})\right] \sin 2\bar{u}\right\} . \quad (179)$$

With the preferred values of k_a and k_i ,

$$U_2 = \frac{1}{48}\left\{(6 - \bar{f} - 13\bar{f}^2) \sin 4\bar{u} + 2 (24 - 106\bar{f} + 93\bar{f}^2) \sin 2\bar{u}\right\} . \quad (180)$$

Again, the general expression for $\dot{U}_2$ is given by combining (153) with the second-order component of $\Delta\bar{n}$, given by (174); thus

$$\dot{U}_2 = \frac{1}{48}\bar{n}\left[(432 - 1052\bar{f} + 637\bar{f}^2) - 24(3 - 4\bar{f})(4k_a + 3k_n) - 192k_i\bar{f}(1 - \bar{f}) + 48k_{2n}\right] . \quad \ldots\ldots (181)$$

When $\dot{U}_1$ is zero, through k_n having its preferred value, it is obviously desirable for $\dot{U}_2$ to be zero as well, and (181) indicates that this requires

$$k_{2n} = -\frac{1}{48}\left\{\bar{f}(100 - 131\bar{f}) - 96k_a(3 - 4\bar{f}) - 192k_i\bar{f}(1 - \bar{f})\right\} . \quad (182)$$

With the preferred values of k_a and k_i , this gives

$$k_{2n} = \frac{1}{48} (288 - 724\bar{f} + 515\bar{f}^2) . \quad (183)$$

The associated value of $\hat{\mu}_2$ is given by (168). Taking k_{2n} as given by (182) we find

$$\hat{\mu}_2 = \frac{1}{12}\left\{(324 - 814\bar{f} + 533\bar{f}^2) - 120k_a (3 - 4\bar{f}) + 72k_a^2 - 96k_i\bar{f} (1 - \bar{f}) - 36k_{2a}\right\} , \quad \ldots\ldots (184)$$

but with the preferred values of k_a, k_i and k_{2a} we get a tremendous simplification, *viz* to

$$\hat{\mu}_2 = \frac{1}{24} \bar{f} (4 - 19\bar{f}) . \quad (185)$$

It may, however, be considered undesirable to tamper with the Kepler relation, in which case $\hat{\mu}_2$ must be zero and we cannot force $\dot{U}_2$ to be zero. But, for (168) to give zero, we require

$$k_{2n} = \tfrac{1}{16}(5\bar{f}^2 + 16k_a^2 - 24k_a k_n + 18k_n^2 - 8k_{2a}) \tag{186}$$

and then (181) gives

$$\dot{\bar{U}}_2 = \tfrac{1}{24}\bar{n}\left\{(216 - 526\bar{f} + 341\bar{f}^2) - 12(3 - 4\bar{f})(4k_a + 3k_n) - 96k_i\bar{f}(1 - \bar{f}) + 72k_a^2 - 108k_a k_n + 81k_n^2 - 36k_{2a}\right\} . \tag{187}$$

Instead of substituting our preferred values of k_a and k_n in (186), it is instructive to set $k_a = k_n = \frac{2}{3}\bar{h}$, with k_{2a} given by (117), *ie* appropriate to $\bar{a} = a'$, since we can then interpret $\bar{n}$ as the exact constant n' . The resulting expression for k_{2n} is given by

$$k_{2n} = -\tfrac{1}{48}\left[(40 - 48\bar{f} - 15\bar{f}^2) - 72k_i\bar{f}(1 - \bar{f})\right] \quad ; \tag{188}$$

with the preferred k_i ,

$$k_{2n} = -\tfrac{1}{48}(40 - 120\bar{f} + 57\bar{f}^2) \quad . \tag{189}$$

5 COMPARISONS WITH OTHER AUTHORS' RESULTS

Of the various papers mentioned in section 1, comparisons will be made only for Refs 22, 25 and 26. The two papers from the French school, one by Berger and Walch[25] and the other by Bretagnon[22], are considered together in section 5.2. Since they both give results for e and ω , rather than ξ and η , suitable formulae for comparison are first derived in section 5.1. The remaining paper, by Kinoshita[26], is considered in section 5.4 after section 5.3 has given results needed for the comparison, namely, for the quantities $\sqrt{a}$ and $\sqrt{a(1 - e^2)}\cos i$.

5.1 Second-order analysis for e and ω

Formulae for second-order perturbations in the elements e and ω are necessarily much more complicated than those in ξ and η since we can no longer confine ourselves to the lowest power of $\bar{e}$ in the expansions. We use our results for ξ and η to derive the formulae we require for e and ω , employing the notation e_1 , e_2 *etc* as in (112) and (113).

The first-order formulae have already been given, *viz* by (77), (78) and (79) for e_1, $\dot{\bar{\omega}}_1$ and ω_1 respectively. It is convenient to split e_1 and ω_1 according to the formulae

$$e_1 = e_{10} + \bar{e}\, e_{11} \tag{190}$$

and

$$\omega_1 = \bar{e}^{-1} (\omega_{10} + \bar{e}\, \omega_{11}) \quad , \tag{191}$$

where e_{10}, e_{11}, ω_{10} and ω_{11} are all independent of $\bar{e}$. Since we shall require two components for both e_2 and ω_2 , it is convenient to split these quantities in a similar way, writing

$$e_2 = \bar{e}^{-1} (e_{20} + \bar{e}\, e_{21}) \tag{192}$$

and

$$\omega_2 = \bar{e}^{-2} (\omega_{20} + \bar{e}\, \omega_{21}) \quad . \tag{193}$$

It will be observed that the singularity underlying the use of e and ω has made e_2 of order $\bar{e}^{-1}$, something not intuitively to be expected perhaps, while making ω_2 of order $\bar{e}^{-2}$.

We shall see, in (205) and (210) following, that e_{20} and ω_{20} in fact say nothing about true second-order perturbations, since they are constrained to algebraic values related only to e_1 and ω_1 . All the information is in e_{21} and ω_{21} , which is why two sets of terms are required for each of e_2 and ω_2 .

We start by substituting in (1) and (2) the expansions

$$e = \bar{e} + \bar{K}\, e_1 + \bar{K}^2\, e_2 \tag{194}$$

and

$$\omega = \bar{\omega} + \bar{K}\, \omega_1 + \bar{K}^2\, e_2 \quad , \tag{195}$$

and relating the resulting expressions to the corresponding expansions of ξ and η . This gives

$$\xi_1 = e_1 \cos \bar{\omega} - \bar{e}\, \omega_1 \sin \bar{\omega} \, , \tag{196}$$

$$\eta_1 = e_1 \sin \bar{\omega} + \bar{e}\, \omega_1 \cos \bar{\omega} \, , \tag{197}$$

$$\xi_2 = \left(e_2 - \tfrac{1}{2}\bar{e}\, \omega_1^2\right) \cos \bar{\omega} - (\bar{e}\, \omega_2 + e_1\, \omega_1) \sin \bar{\omega} \tag{198}$$

and

$$\eta_2 = \left(e_2 - \tfrac{1}{2}\bar{e}\, \omega_1^2\right) \sin \bar{\omega} + (\bar{e}\, \omega_2 + e_1\, \omega_1) \cos \bar{\omega} \quad . \tag{199}$$

The formulae for ξ_2 and ω_2 are of course known, whence E_2 and 0_2 are also known, where

$$E_2 = \xi_2 \cos \bar{\omega} + \eta_2 \sin \bar{\omega} \tag{200}$$

and

$$0_2 = - \xi_2 \sin \bar{\omega} + \eta_2 \cos \bar{\omega} \quad . \tag{201}$$

The definitions involved in (200) and (201) will be extremely useful. The basis for the notation is that the right-hand sides, when the 2-suffices are omitted, give $\bar{e}$ and zero respectively. It is now immediate that

$$E_2 = e_2 - \tfrac{1}{2}\bar{e}\,\omega_1^2 \tag{202}$$

and

$$O_2 = \bar{e}\,\omega_2 + e_1\,\omega_1 \quad . \tag{203}$$

We can now substitute (192) and (191) in (202) to obtain

$$E_2 = \bar{e}^{-1}\,(e_{20} + \bar{e}\,e_{21}) - \tfrac{1}{2}\bar{e}^{-1}\,(\omega_{10} + \bar{e}\,\omega_{11})^2 \quad . \tag{204}$$

Since there are no $\bar{e}^{-1}$ terms in E_2 - for economy of space its expression is not given - it follows from (204) that

$$e_{20} = \tfrac{1}{2}\,\omega_{10}^2 \, , \tag{205}$$

and this is the first 'constraint' referred to earlier. An explicit formula follows at once from (79), from which the ω_{10} and ω_{11} components of ω_1 are defined by (191). We get

$$\begin{aligned} e_{20} = -\tfrac{1}{576}\Big\{&6(24 - 72\bar{f} + 61\bar{f}^2)\cos 2\bar{v} - 240\bar{f}\bar{h}\cos 2\bar{u} + 168\bar{f}h\cos 2(\bar{u}+\bar{v}) \\ &+ 9\bar{f}^2\cos 2(\bar{u}+\bar{\omega}) - 42\bar{f}^2\cos 4\bar{u} + 49\bar{f}^2\cos 2(2\bar{u}+\bar{v}) \\ &+ 72\bar{f}\bar{h}\cos 2\bar{\omega} - 2(72 - 216\bar{f} + 191\bar{f}^2)\Big\} \, . \end{aligned} \tag{206}$$

From identification of the $\bar{e}^{o}$ terms of (204) we see that the other component of e_2 is given by

$$e_{21} = E_2 + \omega_{10}\,\omega_{11} \quad . \tag{207}$$

On evaluation of the right-hand side, we get

$$\begin{aligned} e_{21} = \tfrac{1}{576}\Big\{&9\,[144 - 336\bar{f} + 211\bar{f}^2 - 128k_a\bar{h} - 96k_i\bar{f}(1-\bar{f})]\cos\bar{v} \\ &- 9(16 - 48\bar{f} + 39\bar{f}^2)\cos 3\bar{v} + 6\bar{f}\,[134 - 189\bar{f} - 48k_a + 24k_i(1-\bar{f})]\,\times \\ &\times\cos(\bar{u}+\bar{\omega}) + 8\bar{f}\,[28 - 65\bar{f} - 84k_a + 42k_i(1-\bar{f})]\cos(2\bar{u}+\bar{v}) \\ &- 192\bar{f}\bar{h}\cos(2\bar{u}+3\bar{v}) - 36\bar{f}\bar{h}\cos(\bar{v}-2\bar{\omega}) + 36\bar{f}^2\cos(3\bar{u}+\bar{\omega}) \\ &- 81\bar{f}^2\cos(4\bar{u}+\bar{v}) - 63\bar{f}^2\cos(4\bar{u}+3\bar{v}) + 6k_\omega\,[12\bar{h}\sin\bar{v} \\ &- 3\bar{f}\sin(\bar{u}+\bar{\omega}) + 7\bar{f}\sin(2\bar{u}+\bar{v})]\Big\} \quad . \end{aligned} \tag{208}$$

But the preferred k_ω - see (93) - is a multiple of $\sin 2\bar{\omega}$, causing k_ω terms to combine with earlier terms of (208).

In the same way we can substitute (190), (191), and (193) in (203) to obtain

$$O_2 = \bar{e}^{-1}(\omega_{20} + \bar{e}\,\omega_{21}) + \bar{e}^{-1}(e_{10} + \bar{e}\,e_{11})(\omega_{10} + \bar{e}\,\omega_{11}) \quad , \qquad (209)$$

from which, since O_2 contains no $\bar{e}^{-1}$ terms,

$$\omega_{20} = -e_{10}\,\omega_{10} \quad . \qquad (210)$$

This is the other 'constraint' and it leads to the explicit formula

$$\omega_{20} = -\tfrac{1}{288}\Big\{6(24 - 72\bar{f} + 61\bar{f}^2)\sin 2\bar{v} + 168\bar{f}\bar{h}\sin 2(\bar{u} + \bar{v}) - 9\bar{f}^2 \sin 2(\bar{u} + \bar{\omega}) + 49\bar{f}^2 \sin 2(2\bar{u} + \bar{v}) - 72\bar{f}\bar{h}\sin 2\bar{\omega}\Big\} \quad . \qquad \ldots\ldots (211)$$

Also, from identification of the $\bar{e}^{o}$ terms of (209) we have

$$\omega_{21} = O_2 - e_{10}\,\omega_{11} - e_{11}\,\omega_{10} \quad , \qquad (212)$$

evaluation of which gives

$$\begin{aligned}\omega_{21} = \tfrac{1}{288}\Big\{&6\left[96 - 216\bar{f} + 101\bar{f}^2 - 96k_a\bar{h} - 72k_i\bar{f}(1 - \bar{f})\right]\sin\bar{v} \\ &- 9(16 - 48\bar{f} + 39\bar{f}^2)\sin 3\bar{v} - 18\bar{f}\left[10 - 13\bar{f} - 8k_a + 4k_i(1 - \bar{f})\right] \times \\ &\times \sin(\bar{u} + \bar{\omega}) - 2\bar{f}\left[70 - 59\bar{f} + 168k_a - 84k_i(1 - \bar{f})\right]\sin(2\bar{u} + \bar{v}) \\ &- 192\bar{f}\bar{h}\sin(2\bar{u} + 3\bar{v}) - 36\bar{f}\bar{h}\sin(\bar{v} - 2\bar{\omega}) + 27\bar{f}^2\sin(3\bar{u} + \bar{\omega}) \\ &- 159\bar{f}^2\sin(4\bar{u} + \bar{v}) - 63\bar{f}^2\sin(4\bar{u} + 3\bar{v}) - 3k_e\left[12\bar{h}\sin\bar{v}\right. \\ &\left.- 3\bar{f}\sin(\bar{u} + \bar{\omega}) + 7\bar{f}\sin(2\bar{u} + \bar{v})\right] - 3k_\omega\left[12\bar{h}\cos\bar{v}\right. \\ &\left.+ 3\bar{f}\cos(\bar{u} + \bar{\omega}) + 7\bar{f}\cos(2\bar{u} + \bar{v})\right]\Big\} \quad .\end{aligned} \qquad \ldots\ldots (213)$$

Again the k_e and k_ω terms combine with earlier terms, if preferred values are introduced.

5.2 Comparison with Berger and Walch, and with Bretagnon

Refs 25 and 22 both tabulate large numbers of terms in a compact manner, of which only a small number are involved in the comparisons. A convenient reference

system is desirable, and the following is adopted: BW P.l refers to line l of page P of Ref 25, while B P.pqn refers to the particular line of the appropriate table on page P of Ref 22 that is designated by the given values of p, q and n. Both papers in fact tabulate numerical coefficients against the trigonometric arguments $(p\omega + qM)$ and cover (in a single line of tabulation) more than just the lowest power of e that is relevant here (the powers of e increase in steps of two). The index n relates to powers of f ($= \sin^2 i$) that are allocated different lines of Ref 22 though appearing on the same line of Ref 25.

To get agreement, it will be found that we require $k_a = k_i = 0$ at all times in Ref 25, but various values, that will be stated, for Ref 22. The comparison proceeds through the elements in the usual order, except that e and ω are held over till last.

Starting with a, then, the second-order formula given here as (121) (with $k_{2a} = 0$) is in agreement with BW 122.18 (for $\cos 4\bar{u}$) and (with $k_a = k_i = 0$) BW 122.15 (for $\cos 2\bar{u}$). It also agrees with B 16.444, B 16.222 and B 16.224, so long as k_a is set to $-2\bar{h}$ and k_i to 1.

For i, the second-order formula given here by (127) (with $k_{2i} = 0$) is in agreement with BW 122.47 and BW 122.45. It also agrees with B 22.440, B 22.442, B 22.220 and B 22.222, so long as k_a is set to $-2\bar{h}$ and k_i to -2.

For Ω, we have a secular effect as well as short-periodic terms. The contribution to $\dot{\bar{\Omega}}$, represented here by (133), is the same as is given by BW 110.20, so long as we set k_n (as well as k_a and k_i) to zero; the same value is indicated by Ref 22 through reference to Brouwer's paper (see the second equation on page 394 of Ref 11). For the periodic effect, the formula given here as (134) is in agreement with BW 122.53 and BW 122.51; for agreement with B 22.440, B 22.442, B 22.220 and B 22.222 we must set k_a to $-1\frac{1}{2}\bar{h}$ and k_i to -1.

For U, we have a secular effect that (for an unmodified Kepler relation) is given here by (187), and this agrees with the combination of BW 110.25 and BW 110.30 if k_n and k_{2a} are set to zero. Bretagnon (page 11 of Ref 22) gives the BW 110.30 formula, *ie* for $\dot{\bar{M}}_2$, explicitly, on the grounds that there was an error in Brouwer's paper (last equation on page 393 of Ref 11); as we shall see in sections 5.4 and 9.7, the discrepancy amounts to an implicit use of a non-zero k_{2a} by Brouwer - in giving the explicit formula, Bretagnon

unfortunately introduces two real errors, since he writes J_2 instead of J_2^2 and, less obviously, halves the main term, which should contain 27/4 and not 27/8. For the periodic effect, the formula given here as (179) is in agreement with the combination of BW 123.14 with BW 123.39 and of BW 123.2 with BW 123.27 ; again, the three combinations given by B 28.440, B 28.442 and B 28.444 are also easily verified, and so are the three given by B 28.220, B 28.222 and B 28.224 , so long as we set k_a to $-1\frac{1}{2}\bar{h}$ and k_i to $-\frac{1}{2}$.

We complete the comparisons by looking at e and ω together, calling on results from section 5.1. We have results of two orders of e to check, as has been explained. The lower-order formulae are given here by (206) and (211), and they are in immediate agreement with terms given by BW 122.22, BW 122.27, BW 122.29, BW 122.35, BW 122.37 and BW 122.39 , for e_{20} , and BW 122.56, BW 123.4, BW 123.12 and BW 123.16 , for ω_{20} , constant terms being ignored; the same results are given by B 16. 20, B 16. 22, B 16. 24, B 16.222, B 16.224, B 16.242, B 16.244, B 16.424, B 16.444 and B 16.464 , for e_{20} , and B 28. 20, B 28. 22, B 28. 24, B 28.242, B 28.244, B 28.424 and B 28.464 , for ω_{20} .

The higher-order formulae are given here by (208) and (213), but two corrections are needed before they give agreement with BW 122.21, BW 122.23, BW 122.26, BW 122.28, BW 122.30, BW 122.33, BW 122.36, BW 123.38 and BW 123.40 , for e_{21} , and BW 122.55, BW 122.57, BW 123.1, BW 123.3, BW 123.5, BW 123.9, BW 123.13, BW 123.15 and BW 123.17 , for ω_{21} . First, allowance has to be made for carry-over terms from the lower-order formulae; these arise because the argument is u here but U in Ref 25. Secondly, non-zero values have to be set for k_e and k_ω ; we get the desired agreement if we take* $2\bar{f}\cos 2\bar{\omega} + 8\bar{h}$ for k_e and $-2\bar{f}\sin 2\bar{\omega}$ for k_ω . Ref 22 gives the same results (assuming the same k_e and k_ω), the e_{21} terms being from B 17. 10, B 17. 12, B 17. 14, B 17. 30, B 17. 32, B 17. 34, B 17.212, B 17.214, B 17.2-12, B 17.2-14, B 17.232, , B 17.234, B 17.252, B 17.254, B 17.434, B 17.454 and B 17.474 , but we have to take $k_a = -\frac{1}{2}\bar{h}$ and $k_i = 0$ when $pq = 1$, $k_a = -1\frac{1}{2}\bar{h}$ and $k_i = 2$ when $pq = 21$, and $k_a = -\frac{5}{6}\bar{h}$ and $k_i = \frac{2}{3}$ when $pq = 23$. For the ω_{21} terms agreement is obtained, similarly, from B 29. 10, B 29. 12, B 29. 14, B 29. 30, B 29. 32, B 29. 34, B 29.212, B 29.214, B 29.2-12, B 29.2-14, B 29.232, B 29.234, B 29.252, B 29.254, B 29.434, B 29.454 and B 29.474 , subject to the same assumptions concerning k_a and k_i .

---

* These values of k_e and k_ω are actually $O(e)$ truncations of the k_e and k_ω that are implicit in Refs 22, 25 and 26; they are defined such that the <u>time</u>-averages of δe and $\delta\omega$ are zero, and the full expressions can be obtained from section 9.

This paper has so far not presented a formula for the second-order *secular* perturbation in ω, since this perturbation is $O(Ke)$ in its effect on satellite position (apart from the effect in combination with $\dot{\bar{M}}$, which is allowed for by $\dot{\bar{U}}$), as explained in section 3; $\check{\omega}_2$ is $O(K)$ in fact, as we shall see, whereas ω_{20} contributes an $O(K\,e^{-2})$ term to $\delta\omega$ and ω_{21} contributes an $O(K\,e^{-1})$ term. However, it is perhaps worth deriving, and checking, a formula for $\dot{\bar{\omega}}_2$, so that the secular perturbations are all known to $O(Ke)$ - see also section 9.6.

We can start with Merson's formula for $\dot{\omega}_{sec}$, given by equation (A-27) of Ref 7. This yields

$$\dot{\bar{\omega}}_2 = \tfrac{29}{48}\,\bar{n}\bar{f}\,(4 - 5\bar{f}) \;; \tag{214}$$

the underlying values of the k-constants here are the Kozai values, *viz* $k_a = \bar{h}$, $k_i = 0$ and $k_n = \frac{2}{3}\bar{h}$. From consideration of the (first-order) expression for $\dot{\bar{\omega}}$, given by (78), we can obtain the general contribution of the k's, from which the general formula must be

$$\dot{\bar{\omega}}_2 = \tfrac{1}{48}\bar{n}\left\{(288 - 676\bar{f} + 395\bar{f}^2) - 12(4k_a + 3k_n)(4 - 5\bar{f}) - 120k_i\bar{f}(1 - \bar{f})\right\} \;. \tag{215}$$

We now put $k_a = k_i = k_n = 0$, to obtain a result which agrees with BW 110.25.

In terms of the preferred constants of the present paper we derive

$$\dot{\bar{\omega}}_2 = -\tfrac{1}{48}\bar{n}\,(192 - 476\bar{f} + 325\bar{f}^2) \;. \tag{216}$$

5.3 Second-order analysis for $\sqrt{a}$ and a general result for $a(1 - e^2)\cos^2 i$

If $\sqrt{a}$ is denoted by α, the (second-order) formula for α_2 is easily obtained.

The basic identity is

$$\left(\bar{\alpha} + \bar{K}\,\alpha_1 + \bar{K}^2\,\alpha_2\right)^2 = \bar{a} + \bar{K}\,a_1 + \bar{K}^2\,a_2 \;,$$

which yields the first-order relation

$$2\bar{\alpha}\alpha_1 = a_1$$

and the second-order relation

$$\alpha_1^2 + 2\bar{\alpha}\alpha_2 = a_2 \;.$$

Taking a_1 and a_2 from (65) and (121), we get

$$\alpha_2 = \tfrac{1}{48}\bar{\alpha}\left\{5\bar{f}^2 \cos 4\bar{u} + 4\bar{f}\left[2(5 - 9\bar{f}) - 9k_a + 6k_a(1 - \bar{f})\right] \cos 2\bar{u} - 3(\bar{f}^2 + 2k_a^2 - 8k_{2a})\right\} . \qquad \ldots\ldots (217)$$

We also need to know how the quantity $a(1 - e^2)\cos^2 i$ varies, or rather its square root which we denote by β. Now $a(1 - e^2)$ is simply p, the semi-latus rectum of the osculating elliptic orbit, and it is an elementary result that $\mu p = h^2$, where h is the angular momentum of the satellite's motion relative to the centre of the earth. Clearly h acts normally to the orbital plane, so $h\cos i$ is the angular momentum about the earth's axis. But this is constant for the force associated with any *axi-symmetric* potential, *ie* for any zonal harmonic. Also $h \cos i = \beta$, so our result is that β, like a' and n', is an absolute constant of the motion in the J_2 field.

5.4 Comparison with Kinoshita

Appendix B.1 of Ref 26 lists the second-order short-period perturbations due to J_2^2. There are many terms in each of the five formulae given, and a convenient reference system is to let KB-P.t denote term t of Kinoshita's page B-P.

The five formulae relate, in order, to elements that identify with ξ, η, U, Ω and $\sqrt{\mu a}$ in the notation of the present paper. A sixth formula is unnecessary since Kinoshita's sixth element is $\sqrt{\{\mu a(1 - e^2)\}}\cos i$, and we have seen that this is unperturbed. To get agreement we again require $k_a = k_i = 0$, with k_n also zero for secular quantities. (Like Berger and Walch, and Bretagnon, Kinoshita expresses results in terms of M, rather than v, so that his mean elements are true time averages.)

We start with ξ_2, then, finding that the second-order formula given here as (141) is in agreement with KB-2.8, KB-3.4 and KB-4.1. For η_2, formula (146) here agrees with KB-5.6, KB-6.3 and KB-6.10. For U_2, (179) here agrees with KB-8.7 and KB-9.1; also $\dot{\bar{U}}_2$ given by (187) agrees with the combination of the fourth and fifth formulae of KB-53, so long as we set k_{2a} to $(12 - 30\bar{f} + 25\bar{f}^2)/12$, a value that can be obtained empirically but which is confirmed by the formula for $\sqrt{\mu a}$. (The fifth formula of KB-53, on its own, agrees with $\dot{\bar{\omega}}_2$, as given by (215) with every k set to zero.) For Ω_2, (134) here agrees with KB-10.9 and KB-10.14; also $\dot{\bar{\Omega}}_2$ given by (133)

agrees with the sixth formula of KB-53. For $\sqrt{\mu a}$, (217) here agrees with KB-12.8, KB-12.13 and KB-13.2, if k_{2a} is assigned precisely the 'empirical' value just quoted. (Translated into k_{2n}, by multiplication by $-1\frac{1}{2}$ in accordance with (186), this k_{2a} fully resolves the Brouwer-Bretagnon discord remarked in section 5.2.)

This completes the tie-up between the present results and those of preceding papers, apart from some results for (untruncated) secular and long-periodic perturbations which are derived and verified in section 9.

6 SECOND-ORDER PERTURBATIONS IN POSITION

To obtain perturbations in the cylindrical polar coordinates described in section 2.5, it is essentially a matter of combining existing results, no further integration being required. As a simplification, we throughout set k_a, k_ξ, k_η and k_i to their preferred values, as given by (89), (90), (91) and (99), since this preference was based on the resulting simplicity in the first-order formulae for perturbations in coordinates.

It is convenient to give, in advance, some of the formulae that will be required. Thus, neglecting $O(\bar{e})$ terms in each case,

$$\xi_1 \cos \bar{u} + \eta_1 \sin \bar{u} = \bar{h} + \tfrac{5}{6}\bar{f} \cos 2\bar{u} , \tag{218}$$

$$\xi_1 \sin \bar{u} - \eta_1 \cos \bar{u} = -\tfrac{1}{3}\bar{f} \sin 2\bar{u} , \tag{219}$$

$$\xi_2 \cos \bar{u} + \eta_2 \sin \bar{u} = -\tfrac{1}{36}\bar{f} \left\{3(1 - \bar{f}) \cos 4\bar{u} - 2(10 - 11\bar{f}) \cos 2\bar{u} - 3(1 - 7\bar{f})\right\} , \tag{220}$$

$$\xi_2 \sin \bar{u} - \eta_2 \cos \bar{u} = \tfrac{1}{144} \left\{3\bar{f}(10 - 13\bar{f}) \sin 4\bar{u} + 2(36 - 40\bar{f} - \bar{f}^2) \sin 2\bar{u}\right\} \tag{221}$$

and

$$\left(\xi_1^2 - \eta_1^2\right) \sin 2\bar{u} - 2\xi_1\eta_1 \cos 2\bar{u} = -\tfrac{1}{18}\bar{f} \,(5\bar{f} \sin 4\bar{u} + 12\bar{h} \sin 2\bar{u}) . \tag{222}$$

These formulae follow at once from (70), (72), (142) and (147).

6.1 Perturbation in r

We require formulae for r_1 and r_2, such that

$$r = \bar{r} + \bar{K}\, r_1 + \bar{K}^2\, r_2 , \tag{223}$$

where $\bar{r}$ is derived from mean elements by application of the standard algorithm of section 2.5, whilst r_1 and r_2 are purely periodic.

From (84), it follows that

$$r = \left\{\bar{a} + \bar{K}a_1 + \bar{K}^2 a_2\right\}\left\{1 - (\bar{\xi} + \bar{K}\xi_1 + \bar{K}^2\xi_2)(\cos\bar{U} - \bar{K}U_1 \sin\bar{U}) - (\bar{\eta} + \bar{K}\eta_1 + \bar{K}^2\eta_2)(\sin\bar{U} + \bar{K}U_1 \cos\bar{U}) + \left[(\bar{\xi} + \bar{K}\xi_1)\sin\bar{U} - (\bar{\eta} + \bar{K}\eta_1)\cos\bar{U}\right]^2\right\} , \qquad (224)$$

where only such terms are retained as will actually be needed.

On expanding (224) and comparing with (223), we obtain the first-order formula

$$r_1 = (\bar{r}/\bar{a})\, a_1 - \bar{a}\left[(\xi_1 \cos\bar{U} + \eta_1 \sin\bar{U}) - \bar{e}\sin\bar{M}\,(U_1 + 2\xi_1 \sin\bar{U} - 2\eta_1 \cos\bar{U})\right] , \qquad \ldots\ldots \ (225)$$

which reduces to (85), and hence (88), in view of (53). The formula for r_2, similarly, is

$$r_2 = (\bar{r}/\bar{a})\, a_2 - a_1\,(\xi_1 \cos\bar{U} + \eta_1 \sin\bar{U}) - \bar{a}\left[(\xi_2 \cos\bar{U} + \eta_2 \sin\bar{U}) - (\xi_1 \sin\bar{U} - \eta_1 \cos\bar{U})(\xi_1 \sin\bar{U} - \eta_1 \cos\bar{U} + U_1)\right] . \qquad (226)$$

We can replace $\bar{U}$ by $\bar{u}$ of course, since we are ignoring $O(\bar{K}e)$ perturbations. Then r_2 is a combination of terms given by (122), (65), (218), (220), (219) and (74), the result on reduction being

$$r_2 = -\tfrac{1}{72}\bar{a}\left\{\bar{f}^2 \cos 4\bar{u} + 2\bar{f}\,(26 - 31\bar{f})\cos 2\bar{u} + 72\bar{h} - \bar{f}^2 - 72k_{2a}\right\} . \qquad \ldots\ldots \ (227)$$

Choice of k_{2a}, as foreshadowed by (123), now gives our final result, *viz*

$$r_2 = -\tfrac{1}{72}\bar{a}\bar{f}\left\{\bar{f}\cos 4\bar{u} + 2\,(26 - 31\bar{f})\cos 2\bar{u}\right\} . \qquad (228)$$

6.2 Perturbation in u

We require formulae for u_1 and u_2, such that

$$u = \bar{u} + \bar{K}\,u_1 + \bar{K}^2\,u_2 , \qquad (229)$$

where u_1 and u_2 are purely periodic.

From (94) it follows that

$$
\begin{aligned}
u = {} & (\bar{U} + \bar{K}U_1 + \bar{K}^2 U_2) + 2(\bar{\xi} + \bar{K}\xi_1 + \bar{K}^2\xi_2)(\sin \bar{U} + \bar{K}U_1 \cos \bar{U}) \\
& - 2(\bar{\eta} + \bar{K}\eta_1 + \bar{K}^2\eta_2)(\cos \bar{U} - \bar{K}U_1 \sin \bar{U}) + 1\tfrac{1}{4}\Big\{\big[(\bar{\xi} + \bar{K}\xi_1)^2 \\
& \qquad - (\eta + \bar{K}\eta_1)^2\big] \sin 2\bar{U} - 2(\bar{\xi} + \bar{K}\xi_1)(\bar{\eta} + \bar{K}\eta_1) \cos 2\bar{U}\Big\} , \qquad (230)
\end{aligned}
$$

with only the needed terms retained.

On expanding (230) and comparing with (229), we obtain the first-order formula

$$
\begin{aligned}
u_1 = {} & U_1 + 2\,(\xi_1 \sin \bar{U} - \eta_1 \cos \bar{U}) + \tfrac{1}{2}\bar{e}\,\big[4U_1 \cos \bar{M} + 5\xi_1 \sin(\bar{u} + \bar{v}) \\
& - 5\eta_1 \cos(\bar{u} + \bar{v})\big] , \qquad (231)
\end{aligned}
$$

which reduces to (95) in view of (53). The formula for u_2 , similarly, is

$$
\begin{aligned}
u_2 = {} & U_2 + 2\,(\xi_2 \sin \bar{U} - \eta_2 \cos \bar{U}) + 2U_1\,(\xi_1 \cos \bar{U} + \eta_1 \sin \bar{U}) \\
& + 1\tfrac{1}{4}\,\Big[\big(\xi_1^2 - \eta_1^2\big) \sin 2\bar{U} - 2\xi_1\eta_1 \cos 2\bar{U}\Big] . \qquad (232)
\end{aligned}
$$

We replace $\bar{U}$ by $\bar{u}$ and obtain u_2 as a combination of terms given by (180), (221), (74), (218) and (222), the result being

$$
u_2 = \frac{1}{144}\Big\{(18 - 3\bar{f} - 17\bar{f}^2) \sin 4\bar{u} + 2\,(72 - 170\bar{f} + 97\bar{f}^2) \sin 2\bar{u}\Big\} . \qquad (233)
$$

6.3 Perturbations in r', u' and c

The first-order results, given in section 3, were based on formulae (43) and (44), which are not valid when we proceed to second order. Thus we require a fresh start, based on the formula (45) that effectively defines the cylindrical polar coordinates r', u' and c .

Now $(x \quad y \quad z)^T$ is given by both (41) and (45), so an exact equation for r', u' and c is

$$
\begin{aligned}
& (r' \cos u' \quad r' \sin u' \quad c)^T \\
& \qquad = R_1(\bar{i})\, R_3(\bar{\Omega})\, R_3(-\Omega)\, R_1(-i)\, (r \cos u \quad r \sin u \quad o)^T . \qquad (234)
\end{aligned}
$$

We want to use (234) as source for perturbation formulae that are correct to second order. Now $R_3(\bar{\Omega})\, R_3(-\Omega) = R_3(\bar{\Omega} - \Omega)$ and of course $\Omega - \bar{\Omega} = \delta\Omega$ by

the basic definition (33). We write δ in place of $\delta\Omega$, for convenience, and also write

$$\iota = i - \bar{i} \quad (= \delta i) \quad ,$$

with sine and cosine of i and $\bar{i}$ denoted by s, c, $\bar{s}$ and $\bar{c}$. Then the second-order approximation of (234) is

$$\begin{pmatrix} r' \cos u' \\ r' \sin u' \\ c \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{2}\delta^2 & -c\delta & s\delta \\ \bar{c}\delta & 1 - \frac{1}{2}\iota^2 - \frac{1}{2}c^2\delta^2 & -\iota + \frac{1}{2}sc\delta^2 \\ -\bar{s}\delta & \iota + \frac{1}{2}sc\delta^2 & 1 - \frac{1}{2}\iota^2 - \frac{1}{2}s^2\delta^2 \end{pmatrix} \begin{pmatrix} r \cos u \\ r \sin u \\ 0 \end{pmatrix} .$$

The individual equations may be written as

$$r' \cos u' = r\,(\cos u - c\delta \sin u - \tfrac{1}{2}\delta^2 \cos u) \quad , \tag{235}$$

$$r' \sin u' = r\left(\sin u + \bar{c}\delta \cos u - \tfrac{1}{2}\,[\iota^2 + c^2\delta^2] \sin u\right) \tag{236}$$

and

$$c = r(\iota \sin u - \bar{s}\delta \cos u + \tfrac{1}{2}sc\delta^2 \sin u) \quad . \tag{237}$$

To first order we again see that $r' = r$ and recover the results given by (102) and (100) of section 3.

To second order, we still have identity between r' and r, consequent on our choice of k_i, as remarked at the end of section 2.5. Thus we need not consider r' as a separate quantity, but note that we would have to do so if we were concerned with $O(Ke^2)$ or $O(K^2e)$ perturbations.

We can now obtain the second-order perturbation in u', using either (235) or (236) - as a check we use *both*. From (235) we have

$$\cos u' = \cos u - (\bar{c} - \bar{s}\iota)\,\delta \sin u - \tfrac{1}{2}\delta^2 \cos u \quad , \tag{238}$$

where $\delta = \bar{K}\Omega_1 + \bar{K}^2\Omega_2$ and $\iota = \bar{K}i_1$, while from (236) we have

$$\sin u' = \sin u + \bar{c}\delta \cos u - \tfrac{1}{2}\,(\iota^2 + c^2\delta^2) \sin u \quad . \tag{239}$$

But if υ denotes $u' - u$, we have the Taylor expansions

$$\cos u' = \cos u - \upsilon \sin u - \tfrac{1}{2}\upsilon^2 \cos u$$

and

$$\sin u' = \sin u + \upsilon \cos u - \tfrac{1}{2}\upsilon^2 \sin u \quad .$$

We identify the two expressions for $\cos u'$, and likewise for $\sin u'$, expanding υ according to the formula

$$\upsilon = \bar{K}\,\upsilon_1 + \bar{K}^2\,\upsilon_2 , \tag{240}$$

where υ_1 and υ_2 are required. It is immediate that

$$\upsilon_1 = \bar{c}\,\Omega_1 \tag{241}$$

but we have two formulae for υ_2, *viz*

$$(\upsilon_2 - \bar{c}\Omega_2)\sin u = -\bar{s}i_1\Omega_1 \sin u + \tfrac{1}{2}(\Omega_1^2 - \upsilon_1^2)\cos u \tag{242}$$

from the $\cos u'$ identity, and

$$(\upsilon_2 - \bar{c}\Omega_2)\cos u = \tfrac{1}{2}\,(\upsilon_1^2 - i_1^2 - c^2\Omega_1^2)\sin u \tag{243}$$

from the $\sin u'$ identity. On substitution for υ_1, i_1 and Ω_1, we get the same formula in both cases, *viz*

$$\upsilon_2 - \bar{c}\Omega_2 = -\tfrac{1}{16}\bar{f}\,(1 - \bar{f})(\sin 4\bar{u} + 2\sin 2\bar{u}) . \tag{244}$$

From (136), the formula for Ω_2, we now get

$$\upsilon_2 = -\tfrac{1}{48}\,(1 - \bar{f})\left\{(6 + 5\bar{f})\sin 4\bar{u} + 2(24 - 39\bar{f})\sin 2\bar{u}\right\} . \tag{245}$$

For u_2', it remains to combine (233) and (245), the result being

$$u_2' = -\tfrac{1}{72}\bar{f}\left\{\bar{f}\sin 4\bar{u} - (19 - 20\bar{f})\sin 2\bar{u}\right\} . \tag{246}$$

We finally obtain c_1 and c_2, where

$$c \;(= \delta c) = \bar{K}\,c_1 + \bar{K}^2\,c_2 , \tag{247}$$

from (237). We consider this equation in the form

$$c/r = \bar{K}\,(i_1 \sin u - \bar{s}\,\Omega_1 \cos u) + \bar{K}^2\,(i_2 \sin u - \bar{s}\,\Omega_2 \cos u + \tfrac{1}{2}sc\Omega_1^2 \sin u) . \qquad \ldots\ldots \tag{248}$$

Now from (66) and (68), identifying u and $\bar{u}$ only in the $\bar{e}$ term,

$$i_1 \sin u - \Omega_1 \sin \bar{i} \cos u = \tfrac{1}{2}\sin \bar{i}\cos \bar{i}\left\{\left[\sin u - \sin(2\bar{u} - u)\right] + 1\tfrac{1}{3}\bar{e}\left[2\sin(\bar{u} + \bar{v}) - 3\sin\bar{\omega}\right]\right\} . \qquad \ldots\ldots \tag{249}$$

Thus we have recovered (100), except that there is a 'second-order carry-over' given by the first term of (249), which can be written as $\frac{1}{2}(u - \bar{u}) \sin 2\bar{i} \cos \bar{u}$. The second-order terms now combine to give

$$c_2/r \;=\; i_2 \sin u - \Omega_2 \sin \bar{i} \cos u + \tfrac{1}{4}\Omega_1^2 \sin 2\bar{i} \sin u + \tfrac{1}{2}u_1 \sin 2\bar{i} \cos u \;, \qquad \ldots\ldots (250)$$

which from (128), (136), (68) and (97), replacing u by $\bar{u}$, reduces to

$$c_2 \;=\; -\tfrac{1}{96}\, r \sin 2\bar{i} \left\{ 8\bar{f} \sin 3\bar{u} - (23 - 49\bar{f} - k_{2i}) \sin \bar{u} \right\} \;. \qquad (251)$$

The reason for choosing k_i according to (129) should now be apparent and the final result is simply

$$c_2 \;=\; -\tfrac{1}{12} r\bar{f} \sin 2\bar{i} \sin 3\bar{u} \;. \qquad (252)$$

It is worth remarking that, as with c_1 given by (100), we can replace the factor r by either $\bar{r}$ or $\bar{a}$ if desired. It no longer makes sense to derive an ℓ_2 from u'_2 , however, by multiplying by *any* of these factors.

6.4 Perturbations in instantaneously fixed RLC directions

The basis for the immediately preceding remark (concerning ℓ_2 and u'_2) is that a natural second-order quantity ℓ_2 cannot be defined, since the second coordinate of a cylindrical triple is unalterably angular. (This point has already been made in sections 1 and 2.5.) Thus our fundamental results are for r_2, u'_2 and c_2 , given by (228), (246) and (252) respectively.

For completeness, however, we can obtain formulae for perturbations in three orthogonal directions, *viz* the RLC directions as defined in section 2.5 by instantaneous values of the mean elements. Denoting coordinates in these (fixed) directions as (x', y', z') , we have

$$(x',\; y',\; z') \;=\; \left[r' \cos (u' - \bar{u}),\;\; r' \sin (u' - \bar{u}),\;\; c \right] \;, \qquad (253)$$

where $(\bar{x}', \;\bar{y}', \;\bar{z}') \;=\; (\bar{r},\; 0,\; 0)$.

Introducing the formal expansions of $r'(= r$ to the order required) and u', with corresponding expansions for x', y' and z', we find at once that

$$x'_1 = r_1 \;, \quad y'_1 = \bar{r}u'_1 \;\; (cf\ (47)) \quad \text{and} \quad z'_1 = c_1 \;,$$

whilst the second-order formulae are

$$x_2' = r_2 - \tfrac{1}{2}\,\bar{r}\,u_1'^2 \;, \qquad (254)$$

$$y_2' = \bar{r}\,u_2' + r_1\,u_1' \qquad (255)$$

and

$$z_2' = c_2 \;.$$

(The resemblance of (254) and (255) to familiar formulae of dynamics, immediately clear on replacing r_2 by $\tfrac{1}{2}\ddot{r}$ *etc*, is of course no coincidence.)

Evaluation, based on (88), (102), (228) and (246), gives

$$x_2' = -\tfrac{1}{576}\bar{a}\bar{f}\,\left[7\bar{f}\cos 4\bar{u} + 16\,(26 - 31\bar{f})\cos 2\bar{u} + \bar{f}\right]$$

and

$$y_2' = -\tfrac{1}{144}\bar{a}\bar{f}\,\left[\bar{f}\sin 4\bar{u} - 2\,(19 - 20\bar{f})\sin 2\bar{u}\right] \;.$$

7 NUMERICAL CHECKS

In addition to the formulae checks against Refs 22, 25 and 26, as described in section 5, it was decided to make two numerical checks by comparing a hand-computation of the formulae of this paper with the results of a numerical integration. Both these numerical checks were based on an orbit of zero eccentricity, 63° inclination and 12h period, akin to that of NTS 2 (Ref 27), with a greatly magnified value of J_2 and all other sources of perturbation suppressed.

For the first check, J_2 was set to 0.05 and epoch conditions were chosen to make *mean* eccentricity about 10^{-6}, effectively zero for most purposes, the corresponding *osculating* eccentricity being about 10^{-3}. With $\mu = 398602.0$, an ephemeris was generated by numerical integration, and the values of (osculating) a, $\imath$, Ω, ξ, η, U noted at epoch and after 7 hours, *ie* 210° around the orbit. The changes in the elements, and also in r and u, were compared with values computed from the formulae of sections 4 and 6. The result of this check was entirely satisfactory, the discrepancy for each comparison being of order $\bar{K}^3$ ($\approx 81 \times 10^{-9}$ for the magnified J_2) as expected. In particular, with our usual definition of $\bar{n}$ and $\bar{a}$ such that $\bar{n}$ is given by $\dot{\bar{U}}$ (and hence includes $\dot{\bar{\omega}}$) and k_{2a} is given by (123), we have $\bar{n}^2\bar{a}^3 = 398980.92$, in conformity with the expression

$$\bar{n}^2\bar{a}^3 = \mu\left\{1 + \tfrac{1}{24}\bar{K}\,\left[12(6 - 7\bar{f}) + \bar{K}\bar{f}(4 - 19\bar{f})\right]\right\} \qquad (256)$$

that results from substitutions of (162) and (185) in (158).

The second check was more elaborate and involved use of the NED computer program[27] in a specially modified form. A 12-hour ephemeris, of interval 15 min, was generated by numerical integration, on this occasion with J_2 set to 0.01082628 (ten times the value normally used, but corresponding to a value of K that would be correct for a close earth satellite), $\mu = 398601.3$ and *osculating* eccentricity zero at epoch (mid-point of the 12-hour period), the corresponding *mean* eccentricity being 0.000254. The following 22 parameters of the modified NED were fitted to the 29 points (*ie* 87 coordinate quantities) of the ephemeris: $\bar{a}$, $\bar{e}$, $\bar{i}$, $\bar{\Omega}$, $\bar{\omega}$, $\bar{U}$, $\bar{n}$, $\dot{\bar{\Omega}}$ and 14 coefficients in expressions representing perturbations in r, u' and c . For r , the perturbation terms involved $\cos 2\bar{u}$ and $\cos 4\bar{u}$, as demanded by (88) and (228), together with $\bar{n}t \cos \bar{u}$ and $\bar{n}t \sin \bar{u}$, which are required because NED only allows for $\dot{\bar{\omega}}$ as a component of $\bar{n}$ (see the remarks on suppression of $\dot{\bar{\omega}}$ in section 3). For u' , the perturbation terms involved $\sin 2\bar{u}$, $\sin 4\bar{u}$, $\sin \bar{u}$ and $\cos \bar{u}$, as demanded by (102) and (246), together with $\bar{n}t \cos \bar{u}$ and $\bar{n}t \sin \bar{u}$. For $\bar{c}$, the perturbation terms involved a constant, $\sin 2\bar{u}$, $\cos 2\bar{u}$ and $\sin 3\bar{u}$, as demanded by (100) and (252).

The fitted values for the 22 parameters all agreed with computed values from the formulae of this paper, to within order $\bar{K}^3$ ($\approx 0.82 \times 10^{-9}$ for the magnified J_2 , equivalent to 22 mm at the geocentric distance of NTS 2). The standard deviation for the 87 coordinate residuals was a mere 0.7 mm.

8 COMPLETION OF FIRST-ORDER ANALYSIS FOR POSITION

To complete the analysis of section 3, it is worth giving the full (untruncated) expressions for perturbations in the cylindrical polar coordinates used in this paper. The expressions are quite compact and suggest that the correspondingly complete second-order expressions might not be unduly complicated.

The (accurate) first-order differential expressions for the cylindrical coordinates in terms of elements are[27]

$$\delta r = (r/a)\,\delta a - a\,\delta e \cos v + (a/q)\,e\,\delta M \sin v \ , \tag{257}$$

$$\delta u' = \delta\omega + \delta\Omega \cos i + q^{-1}\,\delta M + (2/q^2)\,(\delta e \sin v + q^{-1}\,e\,\delta M \cos v)$$
$$+ (e/2q^2)\left[\delta e \sin 2v + q^{-1}\,e\,\delta M\,(3 + \cos 2v)\right] \tag{258}$$

and

$$\delta c = r\,(\delta i \sin u - \delta\Omega \sin i \cos u) \ . \tag{259}$$

In these expressions we substitute for δa *etc* from equations (55) *etc*, except that we must introduce suitable values for the k-constants that were omitted from

(55) *etc*. The preferred values of k_e, k_i, k_Ω and k_ω are unchanged from section 3, but k_a , which we now define such that a_1 contains the term $k_a \bar{a}^2/\bar{p}$, has the untruncated form $\bar{h} + \frac{3}{4}\bar{f}\bar{e}^2 \cos 2\bar{\omega}$. (This gives an $\bar{a}$ which is much more convenient than, but only slightly different from, the Kozai semi-major axis, for which the k_a is given by (263); for the absolute constant a' , introduced in section 3, which is incidentally the semi-major axis of Brouwer[11], the untruncated k_a is given by (265), as follows from (64).) We also require k_M , not previously introduced; if this is defined such that "$+k_M$" follows "3f S_4" in the first square brackets of (61), then k_M is required to be identical to k_ω - it will be observed that k_ω and k_M do not exactly cancel in U (they would if U were $q\omega + M$ instead of $\omega + M$), the effective constant in U_1 being $\frac{1}{8}\bar{e}^2 k_\omega/(1 + \bar{q})$.

On this basis, elaboration of (257), (258) and (259) leads to the formulae

$$r_1 = \tfrac{1}{6}\bar{a}\left\{\bar{f}\bar{q}^2 \cos 2\bar{u} + \bar{h}\bar{e}^2 \left[2 - (\bar{r}/\bar{p})(5 - \cos 2\bar{v})\right]\right\} , \tag{260}$$

$$\begin{aligned} u_1' = \tfrac{1}{12}\Big\{\bar{f}\left[\sin 2\bar{u} + 4\bar{e}\sin(\bar{u} + \bar{\omega})\right] + 2\bar{h}\,(\bar{e}/\bar{q}^2)\,\big[4(3 - 4\bar{e}^2)\sin\bar{v} \\ - \bar{e}\sin 2\bar{v} - 4\tfrac{1}{2}\bar{e}\bar{q}^2\,V(\bar{v}, \bar{e})\big]\Big\} \end{aligned} \tag{261}$$

and

$$c_1 = \tfrac{1}{3}r\bar{e}\sin 2\bar{i}\left\{2\sin(\bar{u} + \bar{v}) - 3\sin\bar{\omega} - 1\tfrac{1}{8}\bar{e}\,V(\bar{v}, \bar{e})\cos\bar{u}\right\} , \tag{262}$$

where
$$\begin{aligned} V(v, e) &= -1\tfrac{1}{3}e^{-2}(v - M - 2e\sin v) \\ &= \sin 2v + O(e) \end{aligned}$$

by (50).

Formulae (260) and (261) are respectively compatible with the expressions for dr_s/p and $du_s + d\Omega_s \cos i$ given by Merson on page 8 of Ref 7, if allowance is made for the different values of k_a and k_e employed; Merson was working with the unmodified Kozai elements that are obtained by taking

$$k_a = \tfrac{1}{3}\bar{h}\,(2 + 3\bar{e}^2 + \bar{q}^3) + \tfrac{3}{4}\bar{f}\bar{e}^2 \cos 2\bar{\omega} \tag{263}$$

and

$$k_e = 1\tfrac{1}{3}\bar{h}\left\{5 + 2\bar{q}^2/(1 + \bar{q})\right\} + 3\bar{f}\cos 2\bar{\omega} . \tag{264}$$

Formula (262) effectively combines (but with k_i here unity instead of zero) Merson's expressions for di_s and $d\Omega_s \sin i$.

In using (260) to (262), the first step is the generation of mean position, as usual, and this raises no problem, since untruncated $\bar{\Omega}$ and $\bar{\omega}$ are already

available from $\dot{\hat{\Omega}}_1$ and $\dot{\hat{\omega}}_1$, as given by (67) and (78). If we wish, we can continue to use $\bar{n}$ as the rate of change of $\bar{U}$, as opposed to $\bar{M}$, if we make an $O(\bar{e}^2)$ correction to k_n and $\hat{\mu}_1$, but it is now perhaps more natural to associate $\bar{n}$ with $\bar{M}$. Then $\bar{n}$ no longer includes $\dot{\bar{\omega}}$ and becomes the standard mean motion used by both Kozai[6] and Brouwer[11]; this is actually identical with our n', and the associated k_n (untruncated) is given by

$$k_n = \tfrac{1}{3}\bar{h}\,(2 + 3\bar{e}^2) + \tfrac{3}{4}\bar{f}\bar{e}^2 \cos 2\bar{\omega} \quad , \tag{265}$$

as opposed to the previously 'preferred' value given by (109). There is a consequent change in $\hat{\mu}_1$, which is given by

$$\hat{\mu}_1 = -\,\bar{h}\,(1 - 3\bar{e}^2) \quad ,$$

if k_a is standard (or by $-\bar{h}\bar{q}$ if k_a for the unmodified Kozai semi-major axis is being used), instead of (162). The changes in k_n and $\hat{\mu}_1$ have an inevitable effect on $\hat{\mu}_2$, even if this is still defined so as to make $\dot{\bar{U}}_2$ zero; the new value is given by

$$\hat{\mu}_2 = -\,\tfrac{1}{24}\bar{f}\,(20 - 11\bar{f}) \quad ,$$

instead of (185).

9 ON SECULAR AND LONG-PERIODIC PERTURBATIONS

9.1 General remarks

The original intention was to limit this paper strictly to e-independent J_2^2 perturbations, and it has been seen that the resulting expressions for short-periodic perturbations are very simple. However, for an eccentricity of 0.01, say, a low-altitude satellite experiences long-term perturbations of order $J_2^2\,e$ that, within about a day (since this corresponds to an angular motion of about 100 radians), are of the same order of magnitude as the short-periodic perturbations of order J_2^2 . Hence the paper would not be complete without some reference to these perturbations. Once we go to order $J_2^2\,e$, there is not much difficulty in giving the full formulae, valid for any eccentricity.

The long-term e-independent perturbations are purely secular. When e-dependent terms are considered, however, the long-term perturbations contain long-periodic components, trigonometrically related to the argument of perigee ($\bar{\omega}$), as well as purely secular components. The nature of these perturbations is frequently misunderstood, so some general remarks are offered before the results for the elements a, e, i, Ω, ω and M are given. (There is no need

to go to non-singular elements ξ, η and U , since e^{-1} factors do not appear in the long-term perturbations.)

Suppose a term $K\bar{n}\cos k\bar{\omega}$ (with K not necessarily the J_2-related quantity used in this paper) occurs as a component of the rate of change of some element, ζ say. If $\bar{\omega}$ is constant (as it is for orbits at the critical inclination) we can integrate to get a secular perturbation $K\bar{n}t\cos k\bar{\omega}$. If $\bar{\omega}$ is not constant, but is itself essentially secular with rate $\dot{\bar{\omega}}$, the perturbation can be written in the form

$$\delta\zeta_{\ell p} = \frac{K\bar{n}}{k\dot{\bar{\omega}}} \left[\!\left[\sin k\bar{\omega} \right]\!\right] ,$$

where $[\![\ldots]\!]$ designates "variation over the period of integration", *ie* it is a definite integral. The two results are compatible, since $[\![\sin k\bar{\omega}]\!]/k\dot{\bar{\omega}}$ tends to $t\cos k\bar{\omega}$, over the interval t , as $\dot{\bar{\omega}}$ tends to zero. If, however, $\delta\zeta$ is written with an arbitrary lower limit of integration as with short-periodic perturbations, in particular in the form $\sin k\bar{\omega}/k\dot{\bar{\omega}}$, the perturbation has an apparent singularity for $\dot{\bar{\omega}}$ zero.

As was made clear by Merson[7], trouble is avoided if we keep to definite integrals. Since we are concerned with $k = 2$, we denote by I_c and I_s the time integrals of $\cos 2\bar{\omega}$ and $\sin 2\bar{\omega}$ - in Ref 7 the negatives of these quantities were denoted C_2 and S_2 respectively.

The difficulty is largely a matter of terminology*. The terms 'first-order', 'second-order', *etc*, are unambiguous when used to describe short-periodic and secular perturbations, a second-order (J_2^2) perturbation having the intuitive meaning that the perturbation would have been λ^2 times larger if J_2 had been λ times larger. It is not so simple for a long-periodic perturbation, however, since the frequency of such a perturbation increases by a factor of λ when its amplitude increases, thereby reducing the increase in magnitude of the long-term variation.

It seems most natural to take the order of a long-periodic perturbation from the power of J_2 appearing in the appropriate rate-of-change component, *ie* in $\dot{\zeta}$ for element ζ . With this philosophy, there are no long-periodic

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* The long-standing division of opinion over the problem of the critical inclination is a good example of the semantic confusion related to the meaning of 'order'. As Allan has pointed out[29], the behaviour of certain of the orbital elements over very long periods of time is indeed a libration of amplitude proportional to J_2 , but this does not imply a singularity - the effect arises from *rates of change* that are still only of order J_2^2 .

perturbations of first-order (though J_2 is exceptional here and there are such perturbations for J_3 *etc*). Second-order perturbations arise, and may be expressed in terms of $\bar{K}^2 I_c$ and $\bar{K}^2 I_s$. Since

$$I_c = [\sin 2\bar{\omega}]/2\dot{\bar{\omega}} \quad ,$$

where*

$$\dot{\bar{\omega}} = \tfrac{1}{2}\bar{K}\bar{n}\,(4 - 5\bar{f}) \quad ,$$

$\bar{K}^2 I_c$ behaves as a *first-order* perturbation, and is regularly so described, but in the present paper it will be regarded as still of *second* order. Such perturbations are, after all, not merely less important than first-order secular perturbations - they are also less important than second-order secular perturbations. (Their long-term magnitude is admittedly much greater than for *short*-periodic perturbations of the second order, but this is no more than the usual more-important-if-longer-period effect.)

The above is not the full story, however, as we shall encounter some apparently first-order long-periodic perturbations that are really of the *third* order. The dominant secular perturbations $\dot{\Omega}$ and $\dot{\bar{\omega}}$, given by $-\bar{K}\bar{n}\cos\bar{i}$ and $\tfrac{1}{2}\bar{K}\bar{n}\,(4 - 5\bar{f})$ respectively, are functions of $\bar{e}$ (an argument of $\bar{K}$) and $\bar{i}$, so that they have induced long-periodic variation - thus $\cos\bar{i}$ is equal to an initial (hence constant) value, $\cos\bar{i}_0$, plus a term given by $-\delta i_{\ell p}\sin\bar{i}_0$. The integral of $\bar{K}\,\delta i_{\ell p}$ is itself long-periodic and of third order, but because two integrations are now involved it is apparently first order, and particularly so if written with a factor $\dot{\bar{\omega}}^{-2}$. Since, as we shall see, $\delta e_{\ell p}$ and $\delta i_{\ell p}$ are multiples of I_s , it is natural to introduce the notation II_s for the integral of this quantity, *ie* for the double time integral of $\sin 2\bar{\omega}$.

Two further preparatory points must be made. First, the form each long-periodic perturbation takes is dependent on the expression for the first-order short-periodic perturbation, in that the arbitrary k-constant may be chosen, in different ways, as a function of $\bar{\omega}$. Second, there is an entirely different way of looking at second-order long-term variation, and this was the approach adopted in Refs 2 and 4, in particular. It involves the derivation of each element's variation over a complete revolution of the satellite, to eliminate the short-periodic effect, but now the interpretation of "complete revolution" is crucial -

* Here $\dot{\bar{\omega}}$ is just the first-order secular rate, *ie* $\bar{K}\dot{\bar{\omega}}_1$. We can correct this by the second-order $\bar{K}^2\dot{\bar{\omega}}_2$, but the time derivative of $\bar{\omega}$ is no longer actually constant, now that we consider $\bar{\omega}$ to have long-periodic as well as secular variation. The quantities I_c, I_s and II_s necessarily neglect the long-periodic variation.

the variation from perigee to perigee, for example, is different from the variation from node to node, not because perigee and node are different points but simply because there is a slippage (due to $\dot{\bar{\omega}}$), as between successive perigees and successive nodes.

9.2 Perturbation in a

There is no secular perturbation in the semi-major axis, nor is there any second-order long-periodic variation so long as the ω-dependent part of the 'constant' k_a is specifically $\frac{3}{4}\bar{f}\bar{e}^2 \cos 2\bar{\omega}$. This is the case not only for the three k_a's that have been quoted - the preferred value of this paper, the Brouwer value associated with a' and the Kozai value - but also, it appears, for the k_a implied by two of the papers (the one by Berger and Walch[25] and the one by Kinoshita[26]) considered in section 5. (The value of this k_a is less than the Kozai value by $\bar{h}\bar{q}^3$.) Given an arbitrary k_a , however, we have a long-periodic perturbation given by

$$\delta a_{\ell p} = \bar{K}\bar{a}\bar{q}^{-2} [\![\tfrac{3}{4}\bar{f}\bar{e}^2 \cos 2\bar{\omega} - k_a]\!] . \tag{266}$$

As remarked in section 9.1, there is an apparent long-periodic effect if we only consider the value of a at, say, perigees or ascending nodes. For perigees the effect is given by

$$\delta a_{\ell p} = - \bar{K}^2\bar{n}\bar{a}\bar{q}^{-2} (1 + \bar{e})^3 \bar{f} (4 - 5\bar{f}) I_s ,$$

as indicated by equation (181) of Ref 4.

9.3 Perturbation in e

There is again no secular perturbation, but there is a long-periodic perturbation given by

$$\delta e_{\ell p} = - \tfrac{1}{24}\bar{K}^2\bar{n}\bar{e}\bar{q}^2\bar{f} (14 - 15\bar{f}) I_s + \tfrac{1}{8} \bar{K}\bar{e} [\![3\bar{f} \cos 2\bar{\omega} - k_e]\!] , \tag{267}$$

where the second term vanishes for both our standard k_e and the value implied by use of Kozai elements. However, Refs 25 and 26 involve a value of k_e that is less than Kozai's value by $1\frac{1}{3}\bar{f}\bar{q}^2 (1 + 2\bar{q})(1 + \bar{q})^{-2} \cos 2\bar{\omega}$, whence it follows that, to $O(\bar{e})$, their expression for $\delta e_{\ell p}$ should involve a factor $2(13 - 15\bar{f})$ in place of the factor $(14 - 15\bar{f})$ in the standard expression. This is confirmed by BW 112.15 , in the notation of section 5.2, referring to Ref 25.

9.4 Perturbation in i

A simple relation exists between perturbations in a, e, i, since $p \cos^2 i$ is constant, as remarked in section 5.3. This constancy applies to any potential field that is symmetric about the z-axis, *ie* to the potential associated with any zonal harmonic. Assuming $\delta a_{\ell p}$ to be zero, it follows that

$$\delta i_{\ell p} = -\bar{e}\bar{q}^{-2} \cot \bar{i}\; \delta e_{\ell p} \quad . \tag{268}$$

In fact

$$\delta i_{\ell p} = \tfrac{1}{48}\bar{K}^2 \bar{n} \bar{e}^2 \sin 2\bar{i}\; (14 - 15\bar{f})\; I_s - \tfrac{1}{4}\bar{K} \sin 2\bar{i}\; [k_i] \; , \tag{269}$$

where the second term vanishes for our standard $k_i (= 1)$ and the Kozai $k_i (= 0)$. However, Refs 25 and 26 involve a value of k_i that exceeds Kozai's value by $\frac{1}{3}\bar{e}^2 (1 + 2\bar{q})(1 + \bar{q})^{-2} \cos 2\bar{\omega}$, whence it again follows, to $O(\bar{e})$, that their expression for $\delta i_{\ell p}$ should involve a factor $2(13 - 15\bar{f})$ in place of the factor $(14 - 15\bar{f})$ in the standard expression. This is confirmed by BW 112.18.

9.5 *Perturbation in* Ω

The complete second-order long-term perturbation may be obtained from the following expression, based on Ref 7, for the rate of change of the mean element:

$$\begin{aligned} \frac{d\bar{\Omega}}{dt} = -\bar{K}\bar{n} \cos \bar{i} \Big\{ 1 &+ \tfrac{1}{2}(\dot{\bar{\omega}}/\bar{n})\; \partial k_\Omega/\partial\omega + \tfrac{1}{24}\bar{K}\bar{q}^{-2} \big[4(15 - 19\bar{f}) \\ &+ \bar{e}^2 (4 - 9\bar{f}) - \bar{e}^4 (4 + 5\bar{f}) + \bar{e}^2 \cos 2\bar{\omega} \big((14 - 3\bar{f}) \\ &- 2\bar{e}^2 (7 - 15\bar{f}) \big) - 48\, k_a + 12\bar{e}^2\, k_e - 12\bar{f}\bar{q}^2\, k_i - 36 k_n \big] \Big\} . \end{aligned} \tag{270}$$

We consider the secular component first. With our standard values of k_a, k_e and k_i, and k_n given by (265), we get (*cf* (137) if k_n is from (109))

$$\dot{\bar{\Omega}}_2 = \tfrac{1}{24}\bar{n} \cos \bar{i} \left\{ 4\,(3 - 5\bar{f}) - \bar{e}^2 (4 + 5\bar{f}) \right\} \quad . \tag{271}$$

With the Kozai k's, on the other hand, we get

$$\dot{\bar{\Omega}}_2 = \tfrac{1}{24}\bar{n} \cos i \left\{ 48\bar{h}\bar{q} - 4(9 - 10\bar{f}) - \bar{e}^2 (4 + 5\bar{f}) \right\} \; ,$$

as given (effectively) in Refs 6 and 7. Finally, with the k's of Refs 25 and 26, we get

$$\dot{\bar{\Omega}}_2 = -\tfrac{1}{24}\bar{n} \cos \bar{i} \left\{ 24\bar{h}\bar{q} + 4(9 - 10\bar{f}) + \bar{e}^2 (4 + 5\bar{f}) \right\} \; ,$$

in agreement with BW 110.20-24 and with the sixth formula of KB-53 (notation of section 5.4, in reference to Ref 26) - since Ref 25 uses $(R/\bar{a})^4$ rather than $(R/\bar{p})^4$ as (effectively) a factor of $\bar{K}$, the bracketed expression here must be multiplied by $\bar{q}^{-8}$ before agreement is evident.

Turning to the long-periodic component, we can write the basic second-order perturbation as

$$\delta\Omega_{\ell p(b)} = -\tfrac{1}{12}\bar{K}^2\bar{n}\bar{e}^2 (7 - 15\bar{f}) \cos \bar{i}\ I_c - \tfrac{1}{2}\bar{K} \cos \bar{i}\ [\![k_\Omega]\!] + \tfrac{1}{8}\bar{K}^2\bar{n}\bar{q}^{-2} \cos \bar{i}\ \times$$
$$\times \int\Big\{4\big[4\ (k_a)_\omega - 3\bar{f}\bar{e}^2 \cos 2\bar{\omega}\big] - 4\bar{e}^2\ \big[(k_e)_\omega - 3\bar{f} \cos 2\bar{\omega}\big]$$
$$+ 4\bar{f}\bar{q}^2\ (k_i)_\omega + 3\big[4(k_n)_\omega - 3\bar{f}\bar{e}^2 \cos 2\bar{\omega}\big]\Big\}\, dt\ , \qquad (272)$$

where $(k_a)_\omega$ denotes the ω-dependent (long-periodic) part of k_a, *etc.* But in addition we now have an induced (third-order) perturbation, as explained in section 9.1. On the assumption that the first-order secular perturbation is computed from

$$\dot{\bar{\Omega}}_1 = -\bar{K}_0\ \bar{n}_0 \cos \bar{i}_0\ ,$$

where zero-suffices have been added to indicate that we use *initial* values of the mean elements, our additional perturbation is given by

$$\delta\Omega_{\ell p(ind)} = -\bar{K}\bar{n} \cos \bar{i} \left[4\bar{e}\bar{q}^{-2}\int\delta e_{\ell p}\ dt - \tan \bar{i} \int\delta i_{\ell p}\ dt\right]\ ;$$

from this, using (268), we may write

$$\delta\Omega_{\ell p(ind)} = -5\bar{K}\bar{n}\bar{e}\bar{q}^{-2} \cos \bar{i} \int\delta e_{\ell p}\ dt\ .$$

Hence our standard expression for the combination of $\delta\Omega_{\ell p(b)}$ and $\delta\Omega_{\ell p(ind)}$ is given by

$$\delta\Omega_{\ell p} = -\tfrac{1}{24}\bar{K}^2\bar{n}\bar{e}^2 \cos \bar{i} \left\{2(7 - 15\bar{f})\ I_c - 5\bar{K}\bar{n}\bar{f}\ (14 - 15\bar{f})\ II_s\right\}\ . \qquad (273)$$

Use of the k's of Refs 25 and 26, on the other hand, including $\tfrac{1}{3}\bar{e}^2(1 + 2\bar{q})(1 + \bar{q})^{-2} \sin 2\bar{\omega}$ for k_Ω, leads to

$$\delta\Omega_{\ell p} = -\tfrac{1}{24}\bar{K}^2\bar{n}\bar{e}^2 \cos \bar{i} \left\{\left[2(13 - 30\bar{f}) + O(\bar{e}^2)\right] I_c - 10\bar{K}\bar{n}\bar{f}\ (13 - 15\bar{f})\ II_s\right\}\ ;$$

if (by taking indefinite integrals instead of evaluating I_c and II_s - see the discussion in section 9.1), we interpret this as a first-order perturbation, we get

$$-\frac{\bar{K}\,\bar{e}^2 \cos\bar{i}\;\;(52 - 120\bar{f} + 75\bar{f}^2)}{12\,(4 - 5\bar{f})^2}\sin 2\bar{\omega} + O(\bar{e}^4) \quad ,$$

which tallies with BW 112.20 .

9.6 Perturbation in ω

The complete second-order long-term perturbations may be obtained from the following expression, based on Ref 7, for the rate of change of the mean element:

$$\frac{d\bar{\omega}}{dt} = \tfrac{1}{2}\bar{K}\bar{n}\Big\{(4 - 5\bar{f}) - \tfrac{1}{4}(\dot{\bar{\omega}}/\bar{n})\ \partial k_\omega/\partial\omega + \tfrac{1}{48}\bar{K}\bar{q}^{-2}\Big[2(288 - 676\bar{f} + 395\bar{f}^2)$$
$$- \bar{e}^2(40 + 4\bar{f} - 65\bar{f}^2) - \bar{e}^4(56 - 36\bar{f} - 45\bar{f}^2) - 2\cos 2\bar{\omega}\Big(20(5 - 6\bar{f})$$
$$- \bar{e}^2(28 - 4\bar{f} + 15\bar{f}^2) + \bar{e}^4(28 - 158\bar{f} + 135\bar{f}^2)\Big) - 24(4 - 5\bar{f}) \times$$
$$\times (4k_a - \bar{e}^2k_e + 3k_n) - 240\bar{f}(1 - \bar{f})\bar{q}^2k_i\Big]\Big\} \quad . \tag{274}$$

We consider the secular component first. With our standard values of k_a, k_e and k_i , and k_n given by (265), we get (*cf* (216) if k_n is from (109))

$$\dot{\bar{\omega}}_2 = -\tfrac{1}{96}\bar{n}\left\{2\bar{f}\ (4 + 25\bar{f}) - \bar{e}^2\ (56 - 36\bar{f} - 45\bar{f}^2)\right\} \quad . \tag{275}$$

With the Kozai k's , on the other hand, we get

$$\dot{\bar{\omega}}_2 = -\tfrac{1}{96}\bar{n}\left\{96\bar{h}\bar{q}\ (4 - 5\bar{f}) - 2\ (192 - 412\bar{f} + 215\bar{f}) - \bar{e}^2\ (56 - 36\bar{f} - 45\bar{f}^2)\right\} \quad ,$$

as given (effectively) in Refs 6 and 7. Finally, with the k's of Refs 25 and 26, we get

$$\dot{\bar{\omega}}_2 = \tfrac{1}{96}\bar{n}\left\{48\bar{h}\bar{q}\ (4 - 5\bar{f}) + 2(192 - 412\bar{f} + 215\bar{f}^2) + \bar{e}^2\ (56 - 36\bar{f} - 45\bar{f}^2)\right\} \quad ,$$

in agreement with BW 110.25-29 and with the fifth formula of KB-53; as with $\dot{\Omega}_2$, the bracketed expression must be multiplied by $\bar{q}^{-8}$ to make agreement with Ref 25 evident.

Turning to the long-periodic component, we can write the <u>basic</u> second-order perturbation as

$$\delta\omega_{\ell p(b)} = -\tfrac{1}{48}\bar{K}^2\bar{n}\left\{2\bar{f}\ (14 - 15\bar{f}) - \bar{e}^2\ (28 - 158\bar{f} + 135\bar{f}^2)\right\} I_c$$
$$- \tfrac{1}{8}\bar{K}\ \big[\!\big[k_\omega + 3\bar{f}\sin\bar{\omega}\big]\!\big] - \tfrac{1}{16}\bar{K}^2\bar{n}\bar{q}^{-2}\int\Big\{(4 - 5\bar{f}) \times$$
$$\times \Big(4\big[4(k_a)_\omega - 3\bar{f}\bar{e}^2\cos 2\bar{\omega}\big] - 4\bar{e}^2\ \big[(k_e)_\omega - 3\bar{f}\cos 2\bar{\omega}\big]$$
$$+ 3\big[4(k_n)_\omega - 3\bar{f}\bar{e}^2\cos 2\bar{\omega}\big]\Big) + 40\bar{f}(1 - \bar{f})\bar{q}^2(k_i)_\omega\Big\}\,dt \quad , \tag{276}$$

using the same notation as for $\delta\Omega_{\ell p(b)}$. We also have an induced perturbation that arises from computation of the first-order secular perturbation using

$$\dot{\bar{\omega}}_1 = \tfrac{1}{2}\bar{K}_0\,\bar{n}_0\,(4 - 5\bar{f}_0) \quad .$$

This additional perturbation is given by

$$\delta\omega_{\ell p(ind)} = \bar{K}\bar{n}\left[2\bar{e}\bar{q}^{-2}\,(4 - 5\bar{f})\int\delta e_{\ell p}\,dt - 5\sin\bar{i}\cos\bar{i}\int\delta i_{\ell p}\,dt\right] \quad ;$$

this gives, using (268),

$$\delta\omega_{\ell p(ind)} = \bar{K}\bar{n}\bar{e}\bar{q}^{-2}\,(13 - 15\bar{f})\int\delta e_{\ell p}\,dt \quad .$$

Hence our standard expression for the combination of $\delta\omega_{\ell p(b)}$ and $\delta\omega_{\ell p(ind)}$ is given by

$$\delta\omega_{\ell p} = -\tfrac{1}{48}\bar{K}^2\bar{n}\left\{\left[2\bar{f}\,(14 - 15\bar{f}) - \bar{e}^2\,(28 - 158\bar{f} + 135\bar{f}^2\right]\,I_c + 2\bar{K}\bar{n}\bar{e}^2\bar{f}\,(13 - 15\bar{f})(14 - 15\bar{f})\,II_s\right\} \quad . \tag{277}$$

Use of the k's of Refs 25 and 26, on the other hand, including $-\tfrac{1}{3}\left[3\bar{f} + 4(1 + 2\bar{q})(1 + \bar{q})^{-2}(\bar{f} + \bar{e}^2 - 2\bar{e}^2\bar{f})\right]\sin 2\bar{\omega}$ for k_ω , leads to

$$\delta\omega_{\ell p} = -\tfrac{1}{24}\bar{K}^2\bar{n}\left\{\left[2\bar{f}\,(13 - 15\bar{f}) - \bar{e}^2\,(26 - 155\bar{f} + 140\bar{f}^2) + O(\bar{e}^4)\right]\,I_c + 2\bar{K}\bar{n}\bar{e}^2\bar{f}\,(13 - 15\bar{f})^2\,II_s\right\} \quad .$$

If we take the apparent first-order perturbation that results, we get

$$-\frac{\bar{K}\sin 2\bar{\omega}}{24\,(4 - 5\bar{f})^2}\left\{2\bar{f}\,(52 - 125\bar{f} + 75\bar{f}^2) - \bar{e}^2\,(104 - 412\bar{f} + 555\bar{f}^2 - 250\bar{f}^3) + O(\bar{e}^4)\right\} \quad ;$$

this conforms with BW 112.23-24 if we remember to replace $\bar{K}$ by $\bar{K}\bar{q}^4(1 - \bar{e}^2)^{-2}$ so that the coefficient of $\bar{e}^2$ changes to $-\,(104 - 620\bar{f} + 1055\bar{f}^2 - 550\bar{f}^3)$.

9.7 Perturbation in M

Merson[7] has shown that the rate of change of $\bar{M}$ is given by

$$\frac{d\bar{M}}{dt} = n' - \tfrac{1}{48}\bar{K}^2\bar{n}\bar{q}^3\left\{(8 - 8\bar{f} - 5\bar{f}^2) - 2\bar{f}\,(14 - 15\bar{f})\cos 2\bar{\omega}\right\} + \tfrac{1}{8}\bar{K}\bar{q}\dot{\bar{\omega}}\,(\partial k_M/\partial\omega + 6\bar{f}\cos 2\bar{\omega}) \quad , \tag{278}$$

where the final term has been added here to cover the possibility of a non-standard value of k_M. This expression is 'standard' from the point of view of the present paper.

To generalize from n', which is the absolute constant introduced in section 3, to an arbitrary $\bar{n}$ would require a deeper analysis than will be embarked upon here, since it would involve the derivation of the values of k_{2n} (see section 4.8) appropriate to n' and the selected $\bar{n}$. However, the difficulty is not too great in going from n' to the $\bar{n}$ of Ref 25, so long as it is taken only to $O(e)$. The reward for this is an overall check between Refs 7 and 25 and the formula for k_{2a} in section 4.1

The values of k_{2n}, appropriate to n' and $\bar{n}$ (the $\bar{n}$ of Ref 25 will be understood in what follows), can be derived from the values of k_{2a}, with $\hat{\mu}_1$ and $\hat{\mu}_2$ (see section 4.8) taken as zero in both cases. Thus (161) gives $k_n = k_a$ and then (168) gives

$$k_{2n} = \tfrac{1}{16} (5\bar{f}^2 + 10k_a^2 - 8k_{2a}) \quad .$$

For n' (given by $n'^2 a'^3 = \mu$) we have (untruncated) $k_a = \tfrac{2}{3}\bar{h}(1 + 1\tfrac{1}{2}\bar{e}^2) + {} + \tfrac{3}{4}\bar{f}\bar{e}^2 \cos 2\bar{\omega}$, as quoted in section 8, and k_{2a} given, to $O(\bar{e})$, by (117). This leads to what we may designate $k_{2n}(n')$, an expression for which has already been given - see (188). For $\bar{n}$ (which satisfies $\bar{n}^2 \bar{a}^3 = \mu$), on the other hand, both k_a and k_{2a} are $O(\bar{e}^2)$ so that, to $O(\bar{e})$,

$$k_{2n}(\bar{n}) = \tfrac{15}{16}\bar{f}^2 \quad .$$

But the k_{2n}'s are defined such that

$$n' \left\{1 - 1\tfrac{1}{2}K' \, k_n(n') + K'^2 \, k_{2n}(n')\right\}$$

and

$$\bar{n} \left\{1 - 1\tfrac{1}{2}\bar{K} \, k_n(\bar{n}) + \bar{K}^2 \, k_{2n}(\bar{n})\right\}$$

are identical to $O(\bar{e})$, since they represent the same mean value of the osculating mean motion, n. In making the identification we can take k_i to be zero, in (188), since it is $O(\bar{e}^2)$ in Ref 25. Then we obtain, to $O(\bar{e})$,

$$n' = \bar{n} + K' \, n' \, \bar{h} + \tfrac{1}{48}\bar{K}^2 \, \bar{n} \, (40 - 48\bar{f} + 30\bar{f}^2) \quad .$$

But $K' n'$ must be replaced by $\bar{K}\bar{n}$ in the first-order term of this relation, where in fact

$$K' n' = \bar{K}\bar{n}\,(1 + 2\tfrac{1}{3}\bar{K}\bar{h}) + O(\bar{K}^2\,\bar{e}^2) \quad ,$$

since, to $O(\bar{e})$, $K' = \bar{K}(1 + 1\tfrac{1}{3}\bar{K}\bar{h})$ and $n' = \bar{n}(1 + \bar{K}\bar{h})$. Hence we get

$$n' = \bar{n} + \bar{K}\bar{n}\bar{h} + \tfrac{1}{48}\bar{K}^2\,\bar{n}\,(152 - 384\bar{f} + 282\bar{f}^2) + O(\bar{e}^2) \quad . \tag{279}$$

On combining the $\bar{K}^2$ terms in (278) and (279), we see that, residual to the $\bar{n}$ of Ref 25 and the first-order term $\bar{K}\bar{n}\bar{h}$, $d\bar{M}/dt$ contains the secular component

$$\tfrac{1}{48}\bar{K}^2\,\bar{n}\,(144 - 376\bar{f} + 287\bar{f}^2) \quad .$$

This tallies with BW 110.30 (Ref 25), but not with the fourth formula of KB-53 (Ref 26). However, the apparent error in second-order $\dot{\bar{M}}$, as given by Ref 26, has already been remarked (sections 5.2 and 5.4) and is resolved by allowance for an effective non-zero k_{2a} .

The long-periodic perturbation is easy to express in the appropriate form for checking against Ref 25. The appropriate k_M is $-\tfrac{1}{3}\bar{f}\left\{(3 + 2(2 + \bar{e}^2) \times (1 + 2\bar{q})/(1 + \bar{q}^2)\right\}\sin 2\bar{\omega}$, whereupon the last-term of (278) yields

$$\tfrac{1}{48}\bar{K}^2\,\bar{n}\bar{f}\,(4 - 5\bar{f})\left(6 - 7\bar{e}^2 + O(\bar{e}^4)\right)\cos 2\bar{\omega} \quad .$$

This leads to a perturbation of the form

$$\tfrac{1}{24}\bar{K}^2\bar{n}\bar{f}\left\{2(13 - 15\bar{f}) - 5\bar{e}^2\,(7 - 8\bar{f}) + O(\bar{e}^4)\right\} I_c \quad ; \tag{280}$$

this conforms with BW 112.27 and BW 112.28 if we remember to replace $\bar{K}$ by $\bar{K}\bar{q}^{-4}(1 - \bar{e}^2)^{-2}$, so that the coefficient of $\bar{e}^2$ changes to $(17 - 20\bar{f})$, and also note that BW 112.27, 28, 29 and 30 all contain an unnecessary factor, $(4 - 5\bar{f})$, which may be divided out.

10 DISCUSSION AND CONCLUSIONS

The main goal of the paper has been the derivation of the principal (*ie* e-independent) J_2^2 perturbations of a satellite orbit by an elementary approach, and the expression of the results in as compact a set of formulae as possible. The elementary approach, assuming nothing beyond the planetary equations of Lagrange, has inevitably involved some long and tedious algebra, but simple formulae were eventually obtained by expressing short-periodic perturbations in terms of a rotating system of cylindrical polar coordinates.

The introduction of the paper referred to the sophistication of the literature and to the difficulty in picking out the main results from a mass of

mathematics. It is therefore necessary, to avoid the same charge, that the present author summarize his main results, either directly or by reference to the equation numbers. This will first be done on the basis of the assumption made during most of the analysis, namely, that eccentricity is of the same order of magnitude as J_2, so that J_2e^2, as well as J_2^2e, can be neglected; there is also, of course, the tacit understanding that the results are not to be used over such a long period of time as would make J_2^3 perturbations intolerable, as well as the long-term J_2e^2 and J_2^2e perturbations. Secondly, complete first-order results are indicated, permitting considerable relaxation on the smallness of eccentricity. Finally, to provide results that are valid over an extended period, the complete J_2^2 formulae for long-term development are indicated, involving the special treatment of long-periodic perturbations.

The starting point is assumed to be a set of mean elements at epoch, *viz* $\bar{a}_0$, $\bar{e}_0$, $\bar{i}_0$, $\bar{\Omega}_0$, $\bar{\omega}_0$ and $\bar{M}_0$, whence $\bar{\xi}_0$, $\bar{\eta}_0$ and $\bar{U}_0$ by (1), (2) and (3); values of μ, J_2 and R are also assumed, whence also $\bar{K}$ following (21). The actual generation of these mean elements, given osculating elements (or the coordinates of position and velocity), would not be entirely trivial, but if no other recourse were available they could be adjusted iteratively so that the perturbations led back to the osculating elements. The significance of 'mean elements' was discussed in section 2.4, and the paper contains many general formulae which can be evaluated in terms of particular definitions of these elements.

For our first, and most restricted, derivation of satellite position at time t, we start with $\bar{a}$, $\bar{e}$ and $\bar{i}$, unchanged from $\bar{a}_0$, $\bar{e}_0$ and $\bar{i}_0$, and hence derive $\bar{n}$ from the version of Kepler's third law expressed by (256). The values of $\bar{\Omega}$, $\bar{\omega}$ and $\bar{U}$ are given by

$$\bar{\Omega} = \bar{\Omega}_0 - \bar{K}\bar{n} \cos \bar{i} \left\{1 - \tfrac{5}{6}\bar{K} (3 - 4\bar{f})\right\} t , \tag{281}$$

$$\bar{\omega} = \bar{\omega}_0 + \tfrac{1}{2}\bar{K}\bar{n} (4 - 5\bar{f}) t \tag{282}$$

and

$$\bar{U} = \bar{U}_0 + \bar{n}t , \tag{283}$$

where the expressions for $\dot{\bar{\Omega}}_1$, $\dot{\bar{\Omega}}_2$ and $\dot{\bar{\omega}}_1$, given by (67), (137) and (78), have been employed.

The mean elements lead at once, by the standard algorithm of section 2.5, to $\bar{r}$ and $\bar{u}$. Then (88), (102), (100), (228), (246) and (252) can be combined as

$$r = \bar{r} + \tfrac{1}{8}\bar{K}\bar{a}\bar{f}\left\{\cos 2\bar{u} - \tfrac{1}{12}\bar{K}\left[\bar{f}\cos 4\bar{u} + 2(26 - 31\bar{f})\cos 2\bar{u}\right]\right\} , \tag{284}$$

$$u' = \bar{u} + \tfrac{1}{12}\bar{K}\Big\{\bar{f}\sin 2\bar{u} + 4\bar{e}\left[6\bar{h}\sin\bar{v} + \bar{f}\sin(\bar{u} + \bar{\omega})\right]$$
$$- \tfrac{1}{8}\bar{K}\bar{f}\left[\bar{f}\sin 4\bar{u} - (19 - 20\bar{f})\sin 2\bar{u}\right]\Big\} \tag{285}$$

and

$$c = \tfrac{1}{3}\bar{K}\bar{r}\sin 2\bar{i}\left\{\bar{e}\left[2\sin(\bar{u} + \bar{v}) - 3\sin\bar{\omega}\right] - \tfrac{1}{4}\bar{K}\bar{f}\sin 3\bar{u}\right\} , \tag{286}$$

to give the special cylindrical coordinates. Finally, x, y and z are given by (45), r' being identified with r .

Expressions for osculating elements, if these are desired, are necessarily more complicated than (284) to (286). The element ζ is given by

$$\zeta = \bar{\zeta} + \delta\zeta ,$$

where first-order contributions to $\delta\zeta$, for ζ equal to a, i, Ω, ξ, η and U , are given by (65), (66) (68), (70) (72) and (74), and second-order contributions by (122), (128), (136), (142), (147) and (180); the latter involve preferred values of the k-constants employed throughout this paper, the generalized expressions being given by (121), (127), (134), (141), (146) and (179) - correspondingly generalized $\dot{\bar{\Omega}}_2$ and $\dot{\bar{U}}_2$ are given by (133) and (181).

If the first-order effects are to be covered completely, *ie without* e-truncation, it is perhaps more natural to define $\bar{n}$ so that (283) is replaced by

$$\bar{M} = \bar{M}_0 + \bar{n}t , \tag{287}$$

though the continued absence of a second-order component of $\bar{\omega}$ means that the second-order part of $\bar{n}$ still really relates to $\bar{U}$. This implies a change of k_n , with a new version of Kepler's third law, given by

$$\bar{n}^2\,\bar{a}^3 = \mu\left\{1 - \bar{K}\bar{h}\,(1 - 3\bar{e}^2) - \tfrac{1}{24}\bar{K}^2\,\bar{f}\,(20 - 11\bar{f})\right\} \tag{288}$$

instead of by (256), where the second-order component now comes from the last equation of section 8. The change in k_n affects (281), which has to be replaced by

$$\bar{\Omega} = \bar{\Omega}_0 - \bar{K}\bar{n}\cos\bar{i}\left\{1 - \tfrac{1}{8}\bar{K}\,(3 - 5\bar{f})\right\} t , \tag{289}$$

and the first-order components of r, u' and c are now as given by (260) to (262) and not as in (284) to (286).

If the long-term second-order effects are to be represented completely, with an explicit second-order component of $\dot{\bar{\omega}}$ in particular, then to identify $\bar{n}$ with $\dot{\bar{M}}$ we require yet a further version of Kepler's third law, *viz*

$$\bar{n}^2\,\bar{a}^3 = \mu\left\{1 - \bar{K}_0\bar{h}_0\,(1 - 3\bar{e}_0^2) - \tfrac{1}{8}\bar{K}_0^2\bar{f}_0\,(4 - 9\bar{f}_0)\right\} , \tag{290}$$

in which zero-suffices appear because $\bar{e}$ and $\bar{i}$ are no longer constant. The expressions for the long-term variation of the mean elements are as follows:

$$\bar{a} = \bar{a}_0 , \tag{291}$$

$$\bar{e} = \bar{e}_0 - \tfrac{1}{24}\bar{K}_0^2\bar{n}\,\bar{e}_0\bar{q}_0^{-2}\bar{f}_0\,(14 - 15\bar{f}_0)\,I_s , \tag{292}$$

$$\bar{i} = \bar{i}_0 + \tfrac{1}{48}\bar{K}_0^2\bar{n}\,\bar{e}_0^2\,\sin 2\bar{i}_0\,(14 - 15\bar{f}_0)\,I_s , \tag{293}$$

$$\begin{aligned}\bar{\Omega} = \bar{\Omega}_0 &- \bar{K}_0\bar{n}\cos\bar{i}_0\left\{1 - \tfrac{1}{24}\bar{K}_0\left[4(3 - 5\bar{f}_0) - \bar{e}_0^2(4 + 5\bar{f}_0)\right]\right\} t \\ &- \tfrac{1}{24}\bar{K}_0^2\bar{n}\,\bar{e}_0^2\cos\bar{i}_0\left\{2(7 - 15\bar{f}_0)I_c - 5\bar{K}_0\bar{n}\bar{f}_0(14 - 15\bar{f}_0)II_s\right\} ,\end{aligned} \tag{294}$$

$$\begin{aligned}\bar{\omega} = \bar{\omega}_0 &- \tfrac{1}{2}\bar{K}_0\bar{n}\left\{(4 - 5\bar{f}_0) - \tfrac{1}{48}\bar{K}_0\left[2\bar{f}_0(4 + 25\bar{f}_0) - \bar{e}_0^2(56 - 36\bar{f}_0 - 45\bar{f}_0^2)\right]\right\} t \\ &- \tfrac{1}{48}\bar{K}_0^2\bar{n}\left\{2\bar{f}_0(14 - 15\bar{f}_0) - \bar{e}_0^2(28 - 158\bar{f}_0 + 135\bar{f}_0^2)I_c \right. \\ &\qquad \left. + 2\bar{K}_0\bar{n}\bar{e}_0^2\bar{f}_0(13 - 15\bar{f}_0)(14 - 15\bar{f}_0)II_s\right\} ,\end{aligned} \tag{295}$$

and

$$\bar{M} = \bar{M}_0 + \bar{n}t + \tfrac{1}{24}\bar{K}_0^2\bar{n}\,\bar{q}_0^3\bar{f}_0(14 - 15\bar{f}_0)I_c , \tag{296}$$

where I_c, I_s and II_s are the definite integrals of $\cos 2(\bar{\omega}_0 + \dot{\bar{\omega}}t)$, $\sin 2(\bar{\omega}_0 + \dot{\bar{\omega}}_0 t)$ and I_s, respectively.

No $\bar{K}_0^2\bar{e}_0^2$ contribution is required in (290) because we have not taken the short-periodic perturbations, in a in particular, to $O(K^2e)$, let alone $O(K^2e^2)$; hence the second-order constant k_{2a}, introduced in section 4.2, can be regarded as including whatever component is necessary to eliminate any $O(K^2e^2)$ term.

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The perturbations of ~~order J_2^2~~ in the orbital elements of an earth satellite are analysed by an elementary treatment that neglects terms of ~~order $J_2^2 e$~~. The resulting expressions are combined into a triad of cylindrical polar coordinates, defined by a plane of fixed inclination and uniform rotation rate, since this leads to very simple formulae for perturbations in coordinates.

Mean orbital elements are required and are introduced in a general manner involving arbitrary constants. The normal choice of constants is such as to make both first-order and second-order perturbation formulae as compact as possible for the cylindrical coordinates, the second-order results being expressible as the product of $-\frac{1}{32} J_2^2 (R/a)^2 f$ with, for the three coordinates,

$$af \cos 4u + 2a (26 - 31f) \cos 2u, \quad f \sin 4u - (19 - 20f) \sin 2u$$

and

$$6a \sin 2i \sin 3u ,$$

where R is the earth's equatorial radius; a, i and u are the satellite's (mean) semi-major axis, inclination and argument of latitude, and f is $\sin^2 i$.

Other aspects of the J_2 'main problem' are considered.

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